Sub-models for Interactive Unawareness *†

Simon Grant, J. Jude Kline, Patrick O'Callaghan, John Quiggin

31 March 2014

Abstract

We propose a notion of a sub-model for each agent at each state in the Heifetz et al (2006) model of interactive unawareness. Presuming that each agent is fully cognizant of his sub-model causes no difficulty and fully describes his knowledge and his beliefs about the knowledge and awareness of others. In addition by relaxing some of the conditions that Heifetz et al imposed in their framework we can accommodate agents who have mistaken beliefs about the awareness levels of other agents. However, if we require informational consistency and another condition on knowledge events, then false beliefs are precluded and we return to the original Heifetz et al model.

Keywords: Interactive Unawareness, Sub-models

JEL Classification: C70; C72; D80; D82

1. Introduction

An innovative framework that allows for unawareness in interactive settings has been proposed by Heifetz, Meier, and Schipper (2006), hereafter HMS. This approach has also been taken by Galanis (2013). The modeling technique developed by HMS has been very successful in capturing interactive unawareness in a compact way. One model is used to describe the entire situation involving many agents who may have different awareness levels, and different beliefs about the awareness levels of others.

In game theory, it is traditional to presume that the one model is accessible to all the agents, and this model can be the same one used by the analyst. In the context of an extensive game, Luce and Raiffa (1957, p.49) wrote: "Each agent is fully cognizant of the game in extensive form, that is, he is fully aware of the rules of the game and the utility functions of each of the agents." In the context of models more widely used for

*All authors are affiliated with the School of Economics, Unversity of Queensland. The corresponding author is J. Jude Kline (j.kline@uq.edu.au)

†We thank Burkhard Schipper, Aviad Heifetz, and Spyros Galanis for very constructive comments.

game theoretic analysis, Myerson (1991, p. 64) wrote: "In general, whatever model of a game we may choose to study, the methods of game theory compel us to assume that this model must be common knowledge among the agents." Finally, in the context of the model of knowledge upon which the HMS framework is based, Aumann (1976, p.1237) wrote: "Worthy of note is the implicit assumption that the information partitions are common knowledge."

For games with unawareness, presuming that the analyst and the agents share the same model creates some difficulty. If an agent has access to the analyst's model, conceivably she could use some details of the model to compensate for her unawareness. On the other hand, imputing some model to other agents is necessary if an agent is to reason about the awareness and knowledge of those others. The fact that the agent must model the awareness of others, but does not understand the full model, raises the possibility of false beliefs about the awareness of others.

In this paper, we propose a notion of a sub-model which does not compromise the limited awareness levels of the agents[1]. Given an HMS model, which can be viewed as the model of an analyst, we assign to each agent, at each state, a sub-model of this original model. From the analyst's perspective, the events in the agent's sub-model are tightly connected to a subset of those in the full model by a bijection. Intuitively, the only events in the analyst's model of which the agent is aware at this state are precisely those that map to a corresponding event in her sub-model. This allows us to interpret the events in the full model of which the agent is aware in terms of a sub-model that is entirely from the agent's perspective. Hence, the agent can use her sub-model to reason freely about her awareness as well as the awareness of the other players without contradicting any limitations of her awareness in the analyst's model.

HMS imposed a set of conditions on the possibility correspondences of the players. Among other things, these rule out the possibility that one agent is mistaken about the awareness level of another. By relaxing some of those conditions we show the HMS approach can be extended to accommodate mistakes of this type. However, if we require an additional informational consistency condition in the presence of a property on knowledge events satisfied by HMS models, then we are driven back to the full set of HMS conditions which preclude false beliefs.

One motivating example for wanting to relax the HMS conditions to allow for false beliefs involves Simon, a prospective graduate student from Australia, who has been offered admission to Harvard's PhD program in economics with financial aid. In addition, he is offered a Fulbright scholarship that will pay for his airfare, but requires him to return to Australia for two years upon the completion of his PhD. One benefit he

[1] Halpern-Rego (2012) and Grant-Quiggin (2013) model dynamic unawareness by associating a potentially different extensive game with each node in some underlying extensive game. While the setting of the present paper is static, the use of different extensive games to represent different levels of awareness is similar in spirit to our use of sub-models.

perceives by accepting the Fulbright is as a signal to Michele of his intention to return to Australia. Simon believes that Michele is aware of this signal. Unbeknownst to Simon, however, Michele is blissfully unaware that Simon is sending her a signal through his acceptance of the scholarship. In this sense Simon has a false belief about the awareness level of Michele. In the sequel we formally model this in a modified HMS framework for which all the sub-model properties mentioned above, except the informational consistency property, still hold.[2] We then show that if we add the informational consistency property and a property on knowledge events, then our model is ruled out.

The rest of the paper proceeds as follows. In Section 2 we define a generalized notion of HMS models. We give some basic results including the introduction of two new weakenings of the HMS conditions on possibility correspondences and a characterization of a property of knowledge events that is satisfied by HMS models. In Section 3 we discuss sub-models and show a general existence result under some subset of the HMS conditions. We show that our notion of a sub-model satisfies a set of consistency properties. In Section 4 we introduce the property of informational consistency of sub-models with the information set received. We show that this, together with the conditions required for knowledge of an event to be an event implies there can be no false beliefs. Section 5 concludes.

2. Models of Unawareness

We begin with the definition of a model of interactive unawareness as introduced and developed in HMS. A *model* is a quadruple $(\mathcal{L}, r, N, \Pi)$ where:

M1 (***Base Spaces***): $\mathcal{L} \equiv (\mathcal{S}, \preceq)$ is:

- **(M1.1)** a complete lattice,[3] where:
- **(M1.2)** $\mathcal{S}$ is a non-empty collection of non-empty and disjoint *base spaces.*

HMS interpret each base space $S \in \mathcal{S}$ as encoding everything that can be 'expressed' with the vocabulary of a particular 'language'. We write $S \preceq S'$ for $(S, S') \in \preceq$. HMS interpret $S \preceq S'$ to mean that the language corresponding to the base-space S' is at least as expressive as the language corresponding to the base-space S. We will also write $S \prec S'$ when $S \preceq S'$ and $S \neq S'$. Note that S has a greatest base space (corresponding to the most expressive language with the richest vocabulary in the model) which we denote by S^t and a least base space (corresponding to the least expressive language

[2] Galanis (2013) relaxed a different condition in order to describe the mistake of one agent about another's knowledge.

[3] A *complete lattice* is a pair $(\mathcal{S}, \preceq)$ where $\mathcal{S}$ is a set partially ordered by $\preceq$, and each subset B of $\mathcal{S}$ has an infimum and supremum in $\mathcal{S}$.

with the most impoverished vocabulary) which we denote by S^b.[4] We will denote the union of the base spaces $\cup\{S : S \in \mathcal{S}\}$ by Σ. We denote a generic element in Σ by ω. We write $S(\omega)$ for the base space containing ω, that is, $\omega \in S(\omega)$.

For any ordered pair of base spaces $S \preceq S'$ there exists a projection $r_S^{S'}(\cdot)$ from S' to S, where for each state $\omega \in S'$, $r_S^{S'}(\omega)$ corresponds to the restriction of the description of ω to the more limited vocabulary of the less-expressive language associated with S. Formally, we require:

M2 (***Projections***): $r \equiv (r_S^{S'})_{S \preceq S'}$, where each $r_S^{S'} : S' \to S$ is a surjection satisfying:

> **(Identity map when $S = S'$)** $r_S^S(\omega) = \omega$ for all $\omega \in S$ and $S \in \mathcal{S}$;
> **(Projections Commute)** If $S \preceq S' \preceq S''$, then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

M3 (***Agent Set***). $N = \{1, ..., \ell\}$ is a finite set of agents.

The awareness and information of each agent is encoded in his possibility correspondence.

M4 (***Possibility Correspondences***): $\Pi = (\Pi_i)_{i \in N}$ where for each agent $i \in N$, $\Pi_i : \Sigma \to 2^{\Sigma} \backslash \varnothing$ is called the *possibility correspondence* of agent i. The value $\Pi_i(\omega)$ is the *possibility set* for agent i at state ω.

HMS place a number of restrictions on the possibility correspondences. To describe these, they introduced the following notation. For any $\omega \in S'$, and any $S \preceq S'$, we write ω_S for $r_S^{S'}(\omega)$, that is, ω_S is the projection of ω from its base space S' to the (lower) base space S.

For each base space $S \in \mathcal{S}$, we define the set of base spaces up the lattice from S by:

$$g(S) \equiv \{S' \in \mathcal{S} : S \preceq S'\} \tag{2.1}$$

Define the set of extensions of a subset $B \subseteq S \in \mathcal{S}$ over $g(S)$ by:[5]

$$B^{\uparrow} \equiv \bigcup_{S' \in g(S)} (r_S^{S'})^{-1}(B). \tag{2.2}$$

The following are the additional conditions that HMS imposed on the possibility correspondences. For all $S, S', S'' \in \mathcal{S}$, $\omega, \omega' \in \Sigma$ and $i \in N$:

[4] A set B has a *greatest* (*least*) element under the partial order $\preceq$ iff there exists some $a \in B$ such that $b \preceq a$ ($a \preceq b$) for all $b \in B$. Note that since $\preceq$ is a partial order, the greatest (least) element is unique.

[5] Here, $(r_S^{S'})^{-1}(B) = \{\omega \in S' : r_S^{S'}(\omega) \in B\}$.

C (*Confinement*): If $\omega \in S$, then: $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$;

GR (*Generalized Reflexivity*): $\omega \in \Pi_i(\omega)^{\uparrow}$;

ST (*Stationarity*): $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$;

PPA (*Projections Preserve Awareness*) If $\omega \in S'$, $\omega \in \Pi_i(\omega)$ and $S \preceq S'$, then $\omega_S \in \Pi_i(\omega_S)$;

PPI (*Projections Preserve Ignorance*): If $\omega \in S'$ and $S \preceq S'$, then $\Pi_i(\omega)^{\uparrow} \subseteq \Pi_i(\omega_S)^{\uparrow}$;

PPK (*Projections Preserve Knowledge*): If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, then $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$.[6]

We refer the reader to the discussion in HMS (p83) on the role each of them play in guaranteeing the coherence of the knowledge and awareness of individuals as we move down the lattice of base spaces. Let us just note that we can interpret C (confinement) as requiring that all the states an individual considers possible at a given state ω can be expressed with the same vocabulary, namely the vocabulary of the language available to the individual at ω, which cannot be more expressive than the language in which ω is expressed. Although in the sequel we shall consider relaxations of the other conditions, C seems to us to be fundamental for the coherence of a model and its consistency with the associated knowledge and awareness of the individuals therein.[7] So we shall maintain C throughout.

In the presence of C, we may use $S(\Pi_i(\omega))$ to denote the base space containing $\Pi_i(\omega)$. We now present the notions of events, knowledge, awareness, and unawareness as given in HMS.

Definition 2.1 (Events). *An* event $(B^{\uparrow}, S)$ *consists of a subset* $B^{\uparrow}$ *of* Σ *formed from a subset* B *of* S *according to (2.1) and (2.2).*

We denote the set of events by $\mathcal{E}$. Let $(B^{\uparrow}, S) \in \mathcal{E}$ be an event where $B \neq \varnothing$. By the disjointedness of the base spaces assumed in M1.1, the base space S corresponding to $B^{\uparrow}$ is uniquely determined. Hence, we may follow HMS and write $B^{\uparrow}$ instead of $(B^{\uparrow}, S)$ whenever $B \neq \varnothing$. While events defined in this way are clearly subsets of Σ, as HMS point out, even when $(B^{\uparrow}, S)$ is an event, its full complement $(\Sigma \backslash B^{\uparrow}, S)$ may not be. To deal with this problem, HMS use the relative complement $((S \backslash B)^{\uparrow}, S)$ of an event $(S, B^{\uparrow})$, which is an event. We follow their notation of using $\neg B^{\uparrow}$ for the relative complement $(S \backslash B)^{\uparrow}$. Two important properties of events are described in the following lemma which we state without proof.

[6] Here, $(\Pi_i(\omega))_S = \{\omega'_S : \omega' \in \Pi_i(\omega)\}$.

[7] Indeed the fact that it is designated by HMS (p83) as the "(0)" property for possibility correspondences suggests that they felt it was self-evident that this property should be satisfied.

Lemma 2.2 (Properties of Events). *Let $\mathcal{M} = (\mathcal{L}, r, N, \Pi)$ be a model and let $(B^{\uparrow}, S)$ be an event in $\mathcal{E}$. Then:*

(a) $\omega \in B^{\uparrow}$ if and only if $\omega \in S'$ for some $S' \in g(S)$ and $\omega_S \in B$;

(b) $B^{\uparrow} \cup \neg B^{\uparrow} = S^{\uparrow}$.

Part (a) characterizes the set of states that are contained in an event. Part (b) states that the union of an event and its complement do not cover the whole state space Σ. This is true since we use only the relative complement $\neg B^{\uparrow} = (S \backslash B)^{\uparrow}$ for the event $B^{\uparrow}$. It is what allows HMS to escape the impossibility results of Dekel-Lipman-Rustichini (1998).

Knowledge in HMS models follows Aumann (1976). For an event $(B^{\uparrow}, S)$, the subset $K_i(B^{\uparrow})$ of Ω is defined as:

$$K_i(B^{\uparrow}) \equiv \{\omega \in \Sigma : \Pi_i(\omega) \subseteq B^{\uparrow}\}. \tag{2.3}$$

The subset $K_i(B^{\uparrow})$ is regarded as the set of states in Σ where i knows the event $(B^{\uparrow}, S)$. Unawareness of an event is defined to be:

$$U_i(B^{\uparrow}) \equiv \neg K_i(B^{\uparrow}) \cap \neg K_i(\neg K_i(B^{\uparrow}). \tag{2.4}$$

Finally, awareness of the event corresponds to the subset $A_i(B^{\uparrow}) \equiv \neg U_i(B^{\uparrow})$. As was shown in HMS, awareness can be expressed as:

$$A_i(B^{\uparrow}) = K_i(B^{\uparrow}) \cup K_i(\neg K_i(B^{\uparrow})). \tag{2.5}$$

Returning to the motivating example from the introduction, denote Simon as agent 1 and Michele as agent 2. The model consists of two base spaces $S_T = \{\omega_1, \omega_2\} \succ S_2 = \{a\}$ and is depicted in Figure 1. The base space S_T represents the modeler's view in which ω_1 corresponds to the situation where Michele is unaware that Simon is sending her a signal by accepting the scholarship. This is, in fact, the actual state. However, we have additional states to describe Simon's mistaken belief about Michele's awareness. The state ω_2 corresponds to the hypothetical situation in which Michele is aware of the signal. The base space S_0 and its unique state a is used to describe the awareness level of Michele at the actual state ω_1 where she is not aware of the signal and is aware only that Simon has accepted the scholarship. The projections from the states in S_T to the single state in S_2 are obvious. At states in S_T, Simon always perceives his information as being $\{\omega_2\}$, that is, $\Pi_1(\omega_1) = \Pi_1(\omega_2) = \{\omega_2\}$. For Michele, suppose $\Pi_2(\omega_1) = \{a\}$ and $\Pi_2(\omega_2) = \{\omega_2\}$. Naturally, both Simon and Michele perceive $\{a\}$ as their information at a. The possibility correspondences are depicted in Figure 1 by the filled arrows for Simon and the unfilled arrows for Michele.

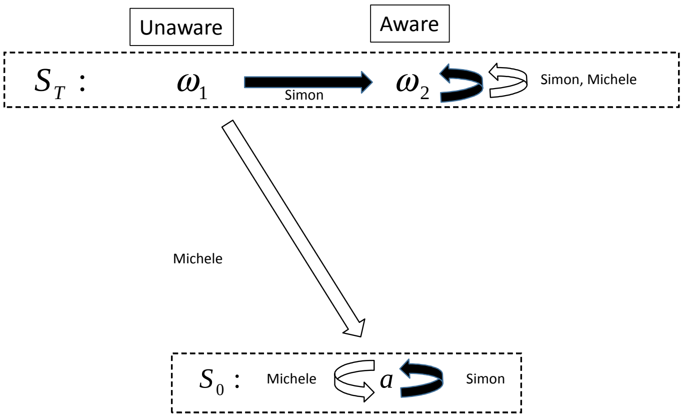

Figure 1: Simon Violates GR

One can readily check that the model satisfies C, ST, PPA, PPI and PPK. However, it violates GR since $\omega_1 \notin \{\omega_2\} = \Pi_1(\omega_1)^{\uparrow}$.

The violation of GR, manifests itself in Simon having a false belief, or more precisely, false knowledge. By Simon having false knowledge, we mean there is an event E for which $K_1(E) \nsubseteq E$. In our example, the event $E = \{\omega_2\}$ describes the situation of Michele being aware of Simon's signal. The event that Simon knows E is $K_1(E) = \{\omega_1, \omega_2\}$. Here, we have a state $\omega_1 \in K_1(E)$, but since $\omega_1 \notin E$, we have $K_1(E) \nsubseteq E$. In other words, at ω_1 Simon 'knows' that Michele is aware of his signal, when in fact she is *not aware* of his signal. In this sense, Simon has false knowledge, which we regard as a false belief.

In order to accommodate such phenomena, in what follows we will drop GR. Formally, we define the property of a model having a false belief as:

FB (*False Belief*) $K_i(E) \nsubseteq E$ for some event $(E, S) \in \mathcal{E}$ and some $i \in N$.

HMS showed (Proposition 2 (iii)) that GR rules out false beliefs, that is, $K_i(E) \subseteq E$ for every event $(E, S) \in \mathcal{E}$ and each agent $i \in N$. FB is actually equivalent to a violation of GR which is stated as the following lemma.

Lemma 2.3 (GR versus FB). *Fix a model $\mathcal{M}$ satisfying C. The following two statements are equivalent:*

1. *The model $\mathcal{M}$ satisfies FB.*
2. *The model $\mathcal{M}$ violates GR.*

Proof. As mentioned above, HMS (Proposition 2 (iii)) proved that GR implies the absence of false beliefs, which is equivalent to ($1 \Rightarrow 2$) of this lemma. Though we assume only C, their proof did not use any of the other conditions, so it applies here. We prove ($2 \Rightarrow 1$). Suppose that $\mathcal{M}$ violates GR, that is, $\omega \notin \Pi_i(\omega)^{\uparrow}$ for some $\omega \in \Sigma$. By C, $\Pi_i(\omega) \subseteq S$ for some $S \in \mathcal{S}$, and $(\Pi_i(\omega)^{\uparrow}, S) \in \mathcal{E}$. Then, $\omega \in K_i(\Pi_i(\omega)^{\uparrow})$ since $\Pi_i(\omega) \subseteq \Pi_i(\omega)^{\uparrow}$, but $\omega \notin \Pi_i(\omega)^{\uparrow}$. $\square$

HMS note that there is some overlap and redundancy between some of the other conditions. In particular, PPA is implied by PPK in the presence of C (see remark 3 in HMS, p83). So, we drop PPA. They also showed (see remark 2 in HMS, p83) that PPI and C imply a condition requiring that as we project from a state down the lattice, the possibility set of an agent associated with the resulting state resides in a lower base space than the original one. Formally,

NIA (*Non-increasing Awareness*) If $\omega \in S, \Pi_i(\omega) \subseteq S'$, $S^{\ell} \preceq S$, and $\Pi_i(\omega_{S^{\ell}}) \subseteq S''$ then $S'' \preceq S'$.

NIA may be interpreted as requiring that the awareness of an agent does not increase as we move down the lattice. Hence, the name. In what follows, we drop GR and relax PPI to NIA. Relaxing PPI, and dropping GR, may come at some cost. One difficulty could be in showing that knowledge of an event (E, S) is itself an S based event. We formally define this property as:

KE (*Knowledge Events*) $(B^{\uparrow}, S) \in \mathcal{E}$ implies $(K_i(B^{\uparrow}), S) \in \mathcal{E}$.

HMS (Proposition 1) showed that KE holds using C, GR, ST, PPI, and PPK. Board et al (2011, Proposition 2.3) showed that KE holds for models satisfying only C, PPI, and PPK. We show that in the presence of C, the property KE is characterized by PPK and NIA.

Proposition 2.4 (KE is equivalent to PPK and NIA). *Fix a model $\mathcal{M}$ satisfying C. The following two statements are equivalent:*

1. *The model $\mathcal{M}$ satisfies KE.*
2. *The model $\mathcal{M}$ satisfies PPK and NIA.*

KE is a fundamental property of HMS models and we will return to it throughout the paper. It secures knowledge as events. Proposition 2.4 shows that in the presence of C, this fundamental property entails PPK and NIA. Since the proof of the proposition is long, we postpone it until the end of this section.

Before it, we show that the HMS characterization of awareness given in their Proposition 3(3), holds in our modified framework. Their characterization is that an agent is aware of an event if and only if he is aware of the base space containing this event.

Proposition 2.5 (Awareness in HMS models). *Fix a model $\mathcal{M}$ satisfying C, ST, NIA and PPK, and let $(B^{\uparrow}, S)$ be an event. Then, for each agent $i \in N$: $A_i(B^{\uparrow}) = K_i(S^{\uparrow})$.*

Proof. To see $A_i(B^{\uparrow}) \subseteq K_i(S^{\uparrow})$, suppose $\omega \in A_i(B^{\uparrow})$. By (2.5), $\omega \in K_i(B^{\uparrow}) \cup K_i(\neg K_i(B^{\uparrow}))$. If $\omega \in K_i(B^{\uparrow})$, then, by (2.3), $\Pi_i(\omega) \subseteq B^{\uparrow}$. By C, $\Pi_i(\omega) \subseteq S'$ for some $S' \in \mathcal{S}$. Since $B^{\uparrow}$ is based on S, it follows that $S' \in g(S)$. Hence, $\Pi_i(\omega) \subseteq S^{\uparrow}$, which by (2.3) implies $\omega \in K_i(S^{\uparrow})$.

If $\omega \in K_i(\neg K_i(B^{\uparrow}))$, then $\Pi_i(\omega) \subseteq \neg K_i(B^{\uparrow})$. By Proposition 2.4, $\neg K_i(B^{\uparrow})$ is based on S. So again, by C, $\Pi_i(\omega) \subseteq S'$ for some $S' \in \mathcal{S}$, and since $\neg K_i(B^{\uparrow})$ is based on S, we have $S' \in g(S)$. Hence, $\Pi_i(\omega) \subseteq S^{\uparrow}$, which by (2.3) implies $\omega \in K_i(S^{\uparrow})$.

Now, let's see that $K_i(S^{\uparrow}) \subseteq A_i(B^{\uparrow})$. Suppose that $\omega \in K_i(S^{\uparrow})$, i.e., $\Pi_i(\omega) \subseteq S^{\uparrow}$. By C, $\omega \in S^{\uparrow}$. By Proposition 2.4, $K_i(B^{\uparrow})$ and $\neg K_i(B^{\uparrow})$ are both S based events. By Lemma 2.2 (b), $\omega \in K_i(B^{\uparrow}) \cup \neg K_i(B^{\uparrow})$. If $\omega \in K_i(B^{\uparrow})$, then by (2.5), $\omega \in A_i(B^{\uparrow})$. If $\omega \in \neg K_i(B^{\uparrow})$, then there is some $\omega' \in \Pi_i(\omega)$ such that $\omega' \notin B^{\uparrow}$. By S, for each $\omega'' \in \Pi_i(\omega)$, $\Pi_i(\omega'') = \Pi_i(\omega)$. Hence, $\omega' \in \Pi_i(\omega'')$ for each $\omega'' \in \Pi_i(\omega)$. This implies that for each $\omega'' \in \Pi_i(\omega)$, $\Pi_i(\omega'') \nsubseteq B^{\uparrow}$, i.e., $\Pi_i(\omega) \subseteq \neg K_i(B^{\uparrow})$. By (2.3) and (2.5), $\omega \in K_i \neg K_i(B^{\uparrow}) \subseteq A_i(B^{\uparrow})$. □

We complete this section with a proof of Proposition 2.4.

Proof of Proposition 2.4. ($2 \Rightarrow 1$): Let $(B^{\uparrow}, S) \in \mathcal{E}$. Then, $B \subseteq S$, and as such, $(\{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow}, S) \in \mathcal{E}$. We will show that $K_i(B^{\uparrow}) = \{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow}$.

First, we show that $K_i(B^{\uparrow}) \subseteq \{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow}$. Suppose $\omega \in K_i(B^{\uparrow})$. Then $\omega \in S''$ for some $S'' \in \mathcal{S}$, and by (2.3), $\Pi_i(\omega) \subseteq B^{\uparrow}$. By C, $\Pi_i(\omega) \subseteq S'$ for some $S' \in \mathcal{S}$ and $S' \preceq S''$. Applying Lemma 2.2 (a) (only-if part) to each $\hat{\omega} \in \Pi_i(\omega)$, we obtain from $\Pi_i(\omega) \subseteq B^{\uparrow}$ that $S \preceq S'$ and $(\Pi_i(\omega))_S \subseteq B$. Using transitivity of $\preceq$ in M1.1, we now have $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$.

It follows by PPK that $\Pi_i(\omega_S) = (\Pi_i(\omega))_S$. Since $(\Pi_i(\omega))_S \subseteq B$, we have $\Pi_i(\omega_S) \subseteq B$, *a fortiori*, $\omega_S \in \{\omega' \in S : \Pi_i(\omega') \subseteq B\}$. Since $S \preceq S''$, it follows by Lemma 2.2 (a) (if part), that $\omega \in \{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow}$.

Next, we show that $\{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow} \subseteq K_i(B^{\uparrow})$. Let $\omega \in \{\omega' \in S : \Pi_i(\omega') \subseteq B\}^{\uparrow}$. By Lemma 2.2 (a) (only-if part), $\omega \in S'$ for some $S' \in \mathcal{S}$ with $S \preceq S'$ and $\Pi_i(\omega_S) \subseteq B \subseteq S$. By C, $\Pi_i(\omega) \subseteq S''$ for some $S'' \preceq S'$. By NIA, $S \preceq S''$

Now, we have $S \preceq S'' \preceq S'$, $\omega \in S'$ and $\Pi_i(\omega) \subseteq S''$. It follows by PPK that $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$. Since $\Pi_i(\omega_S) \subseteq B$, we have $(\Pi_i(\omega))_S \subseteq B$. Since $S \preceq S''$ and $\Pi_i(\omega) \subseteq S''$, we can apply Lemma 2.2 (a) (if-part) to each $\hat{\omega} \in \Pi_i(\omega)$, to obtain $\Pi_i(\omega) \subseteq B^{\uparrow}$. Hence, $\omega \in K_i(B^{\uparrow})$.

($1 \Rightarrow 2$): We prove the contrapositive, that is, if PPK or NIA fails, then KE fails. Observe that PPK can be broken into two conditions:

PPK$^{\supseteq}$ If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, the $(\Pi_i(\omega))_S \supseteq \Pi_i(\omega_S)$.

PPK$^{\subseteq}$ If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, then $(\Pi_i(\omega))_S \subseteq \Pi_i(\omega_S)$.

We break up the proof into three cases: Case 1: $\mathcal{M}$ violates PPK$^{\supseteq}$; Case 2: $\mathcal{M}$ satisfies PPK$^{\supseteq}$, but violates PPK$^{\subseteq}$; Case 3: $\mathcal{M}$ satisfies PPK, but violates NIA. For each case, we will construct an event $(E, S) \in \mathcal{E}$ and show that $(K_i(E), S) \notin \mathcal{E}$.

Case 1: $\mathcal{M}$ violates PPK$^{\supseteq}$.

In this case, we have $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, but $\Pi_i(\omega_S) \not\subseteq (\Pi_i(\omega))_S$. Consider the event $(E, S) = (\Pi_i(\omega))_S^{\uparrow}, S) \in \mathcal{E}$. We will show that $(K_i((\Pi_i(\omega))_S^{\uparrow}), S) \notin \mathcal{E}$. Suppose, on the contrary, that $(K_i((\Pi_i(\omega))_S^{\uparrow}), S) \in \mathcal{E}$. Then, there must be some set B, such that $B \subseteq S$, and $B^{\uparrow} = K_i((\Pi_i(\omega))_S^{\uparrow})$. Since $\Pi_i(\omega) \subseteq (\Pi_i(\omega))_S^{\uparrow}$, it follows that $\omega \in K_i((\Pi_i(\omega))_S^{\uparrow}) = B^{\uparrow}$. By Lemma 2.2 (a), $\omega_S \in B \subseteq B^{\uparrow} = K_i((\Pi_i(\omega))_S^{\uparrow})$. Since $\Pi_i(\omega_S) \not\subseteq (\Pi_i(\omega))_S$, it follows by C that $\omega_S \notin K_i((\Pi_i(\omega))_S^{\uparrow})$, which is a contradiction. Hence, we conclude that $(K_i((\Pi_i(\omega))_S^{\uparrow}), S) \notin \mathcal{E}$.

Case 2 $\mathcal{M}$ satisfies PPK$^{\supseteq}$, but violates PPK$^{\subseteq}$.

In this case, we have $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, but $(\Pi_i(\omega))_S \not\subseteq \Pi_i(\omega_S)$. Consider the event $(E, S) = (\Pi_i(\omega_S)^{\uparrow}, S) \in \mathcal{E}$, which is an event since $\mathcal{M}$ satisfies PPK$^{\supseteq}$. We will show that $(K_i(\Pi_i(\omega_S)^{\uparrow}), S) \notin \mathcal{E}$. Suppose, on the contrary, that $(K_i(\Pi_i(\omega_S)^{\uparrow}), S) \in \mathcal{E}$. Then, there must be some set $B \subseteq S$, and $B^{\uparrow} = K_i(\Pi_i(\omega_S)^{\uparrow})$. Since $\Pi_i(\omega_S) \subseteq \Pi_i(\omega_S)^{\uparrow}$, $\omega_S \in K_i(\Pi_i(\omega_S)^{\uparrow}) = \{\hat{\omega} \in \Sigma : \Pi_i(\hat{\omega}) \subseteq \Pi_i(\omega_S)^{\uparrow}\}$. Since, by assumption, $B^{\uparrow} = K_i(\Pi_i(\omega_S)^{\uparrow})$, it follows by Lemma 2.2 (a) that $\omega \in B^{\uparrow}$. However, since $(\Pi_i(\omega))_S \not\subseteq \Pi_i(\omega_S)$, $\omega \notin K_i(\Pi_i(\omega_S)^{\uparrow})$, a contradiction to $B^{\uparrow} = K_i(\Pi_i(\omega_S)^{\uparrow})$. Hence, we conclude that$(K_i(\Pi_i(\omega_S)^{\uparrow}), S) \notin \mathcal{E}$.

Case 3 $\mathcal{M}$ satisfies PPK, but violates NIA.

In this case we have $\omega \in S, \Pi_i(\omega) \subseteq S'$, $S^{\ell} \preceq S$, and $\Pi_i(\omega_{S^{\ell}}) \subseteq S''$ but $S'' \not\preceq S'$. Consider the event $(E, S'') = (\Pi_i(\omega_{S^{\ell}})^{\uparrow}, S'') \in \mathcal{E}$, and suppose that there is some set $B \subseteq S''$, and $B^{\uparrow} = K_i(\Pi_i(\omega_{S^{\ell}})^{\uparrow})$. Since $\Pi_i(\omega_{S^{\ell}}) \subseteq \Pi_i(\omega_{S^{\ell}})^{\uparrow}$, $\omega_{S^{\ell}} \in K_i(\Pi_i(\omega_{S^{\ell}})^{\uparrow}) = B^{\uparrow}$.

By Lemma 2.2 (a), we find that $\omega \in B^{\uparrow}$. However, since $\Pi_i(\omega) \subseteq S'$ and $S'' \npreceq S'$, $\Pi_i(\omega) \nsubseteq \Pi_i(\omega_{S^\ell})^{\uparrow}$. Hence, $\omega \notin K_i((\Pi_i(\omega_S)^{\uparrow})$, a contradiction. Hence, we conclude that $(K_i(\Pi_i(\omega_S)^{\uparrow}), S) \notin \mathcal{E}$. □

3. Sub-models

Recall the discussion in the introduction about the difficulty of presuming unaware agents are fully cognizant of the model. One of our principal aims is to define a sub-model for each agent which is consistent with the original model, is itself a model, and does not undermine the unawareness of that agent given his information. In this section we introduce a well defined notion of a sub-model of an agent at a state and show that it satisfies the following desiderata.

P1 The sub-model is a model.

P2 The sub-model preserves the events from the original model up to the agent's awareness.

P3 The sub-model preserves knowledge from the original model up to the agent's awareness.

P4 The agent is aware of every event contained in the sub-model.

Property P1 states that the sub-model should inherit the structural properties required for a full model. Properties P2 and P3 may be viewed as a type of consistency between the full model at that state and the agent's sub-model at that state in terms of events and knowledge. They require that the events and knowledge in the sub-model of the agent are consistent with those of the full model up to his awareness. Property P4 requires that an agent's sub-model never has contents that would compromise his awareness.

Let $\mathcal{M} = (\mathcal{L}, r, N, \Pi)$ be an HMS model. We define the sub-model of agent $i \in N$ at $\omega \in \Sigma$ as follows.

Definition 3.1. *The* sub-model $\mathcal{M}^{i,\omega} = (\mathcal{L}^{i,\omega}, r^{i,\omega}, N^{i,\omega}, \Pi^{i,\omega})$ *of agent* $i \in N$ *is defined by:*

R1. (Base Spaces) $\mathcal{L}^{i,\omega} \equiv (\mathcal{S}^{i,\omega}, \preceq^{i,\omega})$ *where:*

(i) $\mathcal{S}^{i,\omega} \equiv \{S' \in \mathcal{S} : S' \preceq S(\Pi_i(\omega))\}$;

(ii) $\preceq^{i,\omega} \equiv \preceq \cap (\mathcal{S}^{i,\omega} \times \mathcal{S}^{i,\omega})$;

R2. (Projections): $r^{i,\omega} \equiv (r_S^{S'})_{S \preceq^{i,\omega} S'}$*;*

R3. (Agent Set) $N^{i,\omega} \equiv N$*;*

R4. (Possibility Correspondences): $\Pi^{i,\omega} = (\Pi_j^{i,\omega})_{j \in N}$ *where for each agent* $j \in N$*,* $\Pi_j^{i,\omega}(\omega') \equiv \Pi_j(\omega')$ *for all* $\omega' \in \Sigma^{i,\omega} \equiv \cup\{S : S \in \mathcal{S}^{i,\omega}\}$*.*

As mentioned in Section 2.2, we will never drop C in this paper. Hence, $S(\Pi_i(\omega))$ and the sub-model $\mathcal{M}^{i,\omega}$ is well defined by R1 to R4 for all $i \in N$ and $\omega \in \Sigma$. We use $\Sigma^{i,\omega}$, $g^{i,\omega}(S)$, $^{\uparrow^{i,\omega}}$, and $\mathcal{E}^{i,\omega}$ to denote, respectively, the union of base spaces, the base spaces at least as expressive as S, the extensions over more expressive base spaces, and the set of events in $\mathcal{M}^{i,\omega}$. Observe that by R1 - R4, the following holds:

$$B^{\uparrow^{i,\omega}} = B^{\uparrow} \cap \Sigma^{i,\omega} \tag{3.1}$$

The next proposition addresses P1 by demonstrating that if we start with a model that satisfies C and possibly other properties, then each sub-model is itself a model inheriting the *same* properties as the original model.

Proposition 3.2 (Sub-models are models). *Fix a model* $\mathcal{M}$ *satisfying C and some set of conditions* $P \subseteq \{GR, ST, PPA, PPI, PPK, NIA\}$*. Then, for any agent* $i \in N$ *and any state* $\omega \in \Sigma$*, the sub-model* $\mathcal{M}^{i,\omega}$ *defined in R1 to R4 is a model satisfying C and every condition in* P*.*

Proof. First, we show that $\mathcal{M}^{i,\omega}$ satisfies M1. It follows by C and M1.2 on $\mathcal{M}$ that M1.2 holds for $\mathcal{L}^{i,\omega} \equiv (\mathcal{S}^{i,\omega}, \preceq^{i,\omega})$. M1.1, namely completeness of $\mathcal{L}^{i,\omega} \equiv (\mathcal{S}^{i,\omega}, \preceq^{i,\omega})$ follows from completeness of $\mathcal{L} \equiv (\mathcal{S}, \preceq)$ and the definition of $\mathcal{S}^{i,\omega}$ as a down-set in R1.(i). Specifically, let $R \subseteq \mathcal{S}^{i,\omega}$. Then, $S(\omega)$ is an upper bound for $\mathcal{S}^{i,\omega}$. By completeness of $\mathcal{L} \equiv (\mathcal{S}, \preceq)$ the set R has a supremum $\sup(R) \in \mathcal{S}$ and a infimum $\inf(R) \in \mathcal{S}$. By the definition of supremum, $\sup(R) \preceq S(\omega)$, and thus $\sup(R)$ is the supremum of R in $\mathcal{S}^{i,\omega}$. Similarly, by the definition of infimum, $\inf(R) \preceq S(\omega)$, and as such $\inf(R)$ is the infimum of R in $\mathcal{S}^{i,\omega}$. M2, M3 and C as well as the conditions in P are inherited from $\mathcal{M}$ by the specifications of R1 to R4. □

Here, the essential condition on the possibility correspondence for a sub-model to be a model is C. The other conditions are not relevant, but are inherited. The lattice structure of the original model is instrumental in ensuring that sub-models are models.

We next show how the events in a sub-model are connected to events in the original model thus addressing P2, P3, and P4. To see this connection, we denote the *set of events of which individual* i *is aware at* ω in the original model by $\mathcal{E}_{A_i,\omega} \equiv \{(B^{\uparrow}, S) \in \mathcal{E} : \omega \in A_i(B^{\uparrow})\}$. As we shall see, the events of which he is aware correspond to the events in his sub-model.

Proposition 3.3 (Consistency of sub-models). *Fix a model $\mathcal{M}$ satisfying C. Then for every agent $i \in N$ and at each state $\omega \in \Sigma$ the sub-model $\mathcal{M}^{i,\omega}$ defined by R1-R4 satisfies the following:*

(a) (preservation of events) The function $f : \mathcal{E}_{A_{i,\omega}} \to \mathcal{E}^{i,\omega}$ defined by $f(B^{\uparrow}, S) = (B^{\uparrow^{i,\omega}}, S)$ is a bijection.

(b) (preservation of knowledge) If in addition $\mathcal{M}$ satisfies NIA and PPK, then: $K_j^{i,\omega}(B^{\uparrow^{i,\omega}}) = K_j(B^{\uparrow}) \cap \Sigma^{i,\omega}$ for each $j \in N$ and each event $(B^{\uparrow^{i,\omega}}, S) \in \mathcal{E}^{i,\omega}$.

(c) (full awareness) If in addition to the properties that hold in (b), $\mathcal{M}$ satisfies ST then: $\Pi_i(\omega) \subseteq A_i^{i,\omega}(B^{\uparrow^{i,\omega}})$ for each event $(B^{\uparrow^{i,\omega}}, S) \in \mathcal{E}^{i,\omega}$.

Proof *(a)* By C on $\mathcal{M}$ and Proposition 3.2, the sub-model $\mathcal{M}^{i,\omega}$ is well defined. First, we show that the function f is an injection. Suppose $(B^{\uparrow}, S)$ and $(C^{\uparrow}, S')$ are two distinct events in $\mathcal{E}_{A_{i,\omega}}$. Then, either $B^{\uparrow} \neq C^{\uparrow}$, or $S \neq S'$. In either case, $(B^{\uparrow^{i,\omega}}, S)$ is distinct from $(C^{\uparrow^{i,\omega}}, S')$. Now, let's see that f is a surjection. Let $(B^{\uparrow^{i,\omega}}, S) \in \mathcal{E}^{i,\omega}$. Then $S \in \mathcal{S}^{i,\omega}$, $(B^{\uparrow}, S) \in \mathcal{E}$ and $f(B^{\uparrow}, S) = (B^{\uparrow^{i,\omega}}, S)$. We need to show that $(B^{\uparrow}, S) \in \mathcal{E}_{A_{i,\omega}}$, which is equivalent to showing that $\omega \in A_i(B^{\uparrow})$. Let S' denote the base space containing $\Pi_i(\omega)$. Since $S \in \mathcal{S}^{i,\omega}$ it follows from R1, that $S \preceq S'$. Hence, $\Pi_i(\omega) \subseteq S' \in S^{\uparrow}$. By (2.3), $\omega \in K_i(S^{\uparrow})$.

(b) By C, NIA and PPK on $\mathcal{M}$, we can use Proposition 2.4 and Proposition 3.2 to deduce that knowledge events are well defined in $\mathcal{M}$ and $\mathcal{M}^{i,\omega}$. Let $\omega' \in K_j^{i,\omega}(B^{\uparrow^{i,\omega}})$. Then $\omega' \in \Sigma^{i,\omega} \subseteq \Sigma$ and $\Pi_j(\omega') \subseteq B^{\uparrow^{i,\omega}} \subseteq B^{\uparrow}$. Hence, $\omega' \in K_j(B^{\uparrow}) \cap \Sigma^{i,\omega}$. Conversely, suppose that $\omega' \in K_j(B^{\uparrow}) \cap \Sigma^{i,\omega}$. Then, $\Pi_j(\omega') \subseteq B^{\uparrow}$ and $\omega' \in \Sigma^{i,\omega}$. By C, $\Pi_j(\omega') \subseteq \Sigma^{i,\omega}$. Since $B^{\uparrow^{i,\omega}} = B^{\uparrow} \cap \Sigma^{i,\omega}$, we have $\Pi_j(\omega') \subseteq B^{\uparrow^{i,\omega}}$. By (2.3), $\omega' \in K_j^{i,\omega}(B^{\uparrow^{i,\omega}})$.

(c) By C, ST, NIA, and PPK on $\mathcal{M}$, we can use Proposition 3.2 to deduce that $\mathcal{M}^{i,\omega}$ also satisfies C, ST, NIA, and PPK. Thus we can apply Proposition 2.5 together with part *(b)* of this proposition, to conclude that $A_i^{i,\omega}(B^{\uparrow^{i,\omega}}) = K_i^{i,\omega}(S^{\uparrow^{i,\omega}})$. Let S' denote the base space containing $\Pi_i(\omega)$. Then, $S' \subseteq S^{\uparrow^{i,\omega}}$. By C, ST and R4, $\Pi_i(\omega') \subseteq S'$ for each $\omega' \in \Pi_i(\omega)$. Hence, $\Pi_i(\omega) \subseteq S^{\uparrow^{i,\omega}}$. By (2.3), $\Pi_i(\omega) \subseteq A_i^{i,\omega}(B^{\uparrow^{i,\omega}}) = K_i^{i,\omega}(S^{\uparrow^{i,\omega}})$. □

Part *(a)* shows that the events of which agent i is aware in $\mathcal{M}$ at ω correspond neatly to the events in his sub-model $\mathcal{M}^{i,\omega}$ and so our model satisfies P2. Part *(b)* shows that knowledge from the original model is preserved in the sub-model, that is P3 holds. Finally, part *(c)* applies to property P4. It states that the agent is fully aware of all events in his sub-model $\mathcal{M}^{i,\omega}$ when we take his local information to be $\Pi_i(\omega)$. By property ST, he perceives the same set of possibilities at each state $\omega' \in \Pi_i(\omega)$. Since the base space containing $\Pi_i(\omega)$ is the top of his sub-model, the events in his sub-model consist only of states that do not exceed his awareness level.

In light of these results, we are able to presume that each agent is fully cognizant of his sub-model without undermining his unawareness. It is worth mentioning that part *(c)* of this theorem states only that agent i is not unaware of any event in his sub-model. It does not, however, prevent another agent j from being unaware of some event contained in i's sub-model. It also allows i to presume he would have been unaware, had some state outside his possibility set $\Pi_i(\omega)$ occurred.

In our analysis, we have not used GR, and thus we can use the motivating example to discuss the results of this section. At the state ω_1, where Michele is not aware of Simon's signal, the only event of which she is aware in the grand model is $\Sigma = \{a, \omega_1, \omega_2\}$. Although one might be tempted to interpret this as implying she is aware of the states ω_1 and ω_2 that are contained in Σ, this is simply not correct. However, her sub-model at ω_1 is one containing just a single base space $S_0 = \{a\}$. So adopting our approach, one can interpret her awareness in the grand model of the event Σ as corresponding to her awareness of the event $\Sigma^{2,\omega_1} = \{a\}$ in her sub-model.

Simon's sub-model at ω_1 is the entire model described in Figure 1. Here, he mistakenly believes Michele has access to the entire model as well, since he thinks ω_2 is the only possible state, and at this state Michele's sub-model would be the entire model. Here we see mistaken beliefs about the other's awareness in action in the HMS framework.

4. Informational Consistency

As we have seen in the previous section, the consistency properties P1 to P4 allow us to interpret the events in the full model in terms of events in an agent's sub-model. This was done in a modified HMS framework that violates GR allowing us to capture mistaken beliefs about awareness. In this section, we consider how an agent could use his possibility set $\Pi_i(\omega)$ to check the consistency with his sub-model $\mathcal{M}^{i,\omega}$. The potential for the agent to conduct such a consistency check provides our motivation for adding a fifth desideratum to the list from section 3.

P5 The sub-model is informationally consistent with that agent's possibility set at that state.

Property P5 ensures that the agent does not find any discrepancy between his sub-model and his possibility set. We formalize this notion as follows.

Definition 4.1 (Sub-model Information Consistency [SMIC]). *A model $\mathcal{M}$ satisfies sub-model information consistency (SMIC) if for each state $\omega \in \Sigma$ and for each agent $i \in N$, the sub-model $\mathcal{M}^{i,\omega}$ satisfies*:

$$\Pi_i(\omega) = \{\omega' \in \Sigma^{i,\omega} : \Pi_i^{i,\omega}(\omega') = \Pi_i(\omega)\} \tag{4.1}$$

In the definition of SMIC, we have intentionally made reference to the agent's sub-model.[8] Here we assume that the information received by agent i at ω comes in two parts, one part is his sub-model $\mathcal{M}^{i,\omega}$ and the other is his possibility set $\Pi_i(\omega)$. Although, in general, the agent need not be aware of ω, we presume that he has access to the contents of $\Pi_i(\omega)$. He can then use his possibility set $\Pi_i(\omega)$ to check if everything meshes with the states in his sub-model. For example, if he were to find some state ω' that is in $\Pi_i(\omega)$, but is not in his sub-model, then he would have encountered an inconsistency between the two parts of his information. More generally, if the set of states in his sub-model where he should receive the possibility set equivalent to $\Pi_i(\omega)$ is different to the possibility set $\Pi_i(\omega)$ he has received, he regards his sub-model $\mathcal{M}^{i,\omega}$ to be informationally inconsistent with his possibility set $\Pi_i(\omega)$.

We use our motivating example to illustrate this consistency check. Recall that Michele's sub-model at ω_1 is the sub-model comprising a single one-element base space $S_0 = \{a\}$ and her possibility set is:

$$\Pi_2^{2,\omega_1}(\omega_1) = \{a\}. \tag{4.2}$$

Now if she checks this possibility set with her sub-model, she will find that the set of states in her sub-model that could have generated this possibility set is:

$$\{\omega' \in \Sigma^{2,\omega_1} : \Pi_2^{2,\omega_1}(\omega') = \{a\}\} = \{a\}. \tag{4.3}$$

Since the contents in (4.2) and (4.3) are the same, the two parts of her information at ω_1 are informationally consistent.

Turning to Simon, recall that his sub-model at ω_1 is the original model. Furthermore, at ω_1 he receives the possibility set:

$$\Pi_1^{1,\omega_1}(\omega_1) = \{\omega_2\}. \tag{4.4}$$

Now, if he checks this possibility set with his sub-model, he will find that the set of states in his sub-model that could have generated this possibility set is:

$$\{\omega' \in \Sigma^{1,\omega_1} : \Pi_1^{1,\omega_1}(\omega') = \{\omega_2\}\} = \{\omega_1, \omega_2\}. \tag{4.5}$$

Since the contents in (4.4) and (4.5) are *not* the same, the two parts of his information at ω_1 are inconsistent. Thus we have a violation of SMIC.

We will show that SMIC is equivalent to the condition ST on the possibility correspondence together with another condition that is a weakening of GR, which is itself a weakening of the well known condition called R for reflexivity. The condition R is equivalent to $K_i(E) \subseteq E$ for all $E \in \mathcal{E}$, in a model with a single base space.

[8] Since, by R4 in the definitiion of a sub-model, $\Pi_i^{i,\omega}(\omega') = \Pi_i(\omega')$, the expression (4.1) defining SMIC could have been written simply in terms of the possibility correpondence $\Pi_i(\cdot)$ of the full model.

R (*Reflexivity*) $\omega \in \Pi_i(\omega)$ for all $\omega \in \Sigma$.

R requires that the possibility set at a state must include that state. Our weakening only requires this inclusion at a state ω when the agent is aware of the base containing $S(\omega)$. Because of this conditional statement which depends on awareness, we have the following name:

RGA (*Reflexivity Given Awareness*) If $\omega \in S$ and $\Pi_i(\omega) \subseteq S$ for some $S \in \mathcal{S}$, then $\omega \in \Pi_i(\omega)$.

The name 'reflexivity given awareness' indicates that reflexivity is required only at a state ω where the agent is aware of the base space containing ω. This follows since knowledge implies awareness and under the conditional part of RGA, the agent knows the base space containing ω.

Our motivating example violates RGA since $\omega_1 \in S^T$ and $\Pi_1(\omega_1) \subseteq S^T$, but $\omega_1 \notin \{\omega_2\} = \Pi_1(\omega_1)$. The next proposition shows that in the presence of C, SMIC is characterized by ST and RGA.

Proposition 4.2 (Informational Consistency). *Fix a model $\mathcal{M}$ satisfying C. The following two statements are equivalent:*

1. *The model $\mathcal{M}$ satisfies SMIC.*
2. *The model $\mathcal{M}$ satisfies RGA and ST.*

Proof ($2 \Rightarrow 1$) Fix $\omega \in \Sigma$ and $i \in N$. First we show $\Pi_i(\omega) \subseteq \{\omega' \in \Sigma^{i,\omega} : \Pi_i(\omega') = \Pi_i(\omega)\}$. Let $\hat{\omega} \in \Pi_i(\omega)$ Then, $\hat{\omega} \in \Sigma^{i,\omega}$. By ST, $\Pi_i(\hat{\omega}) = \Pi_i(\omega)$. Hence, $\hat{\omega} \in \{\omega' \in \Sigma^{i,\omega} : \Pi_i(\omega') = \Pi_i(\omega)\}$. Next we show $\{\omega' \in \Sigma^{i,\omega} : \Pi_i(\omega') = \Pi_i(\omega)\} \subseteq \Pi_i(\omega)$. Let $\hat{\omega} \in \Sigma^{i,\omega}$ such that $\Pi_i(\hat{\omega}) = \Pi_i(\omega)$. By C, $\Pi_i(\hat{\omega}) \subseteq S(\hat{\omega})$. Hence, by RGA, $\hat{\omega} \in \Pi_i(\hat{\omega}) = \Pi_i(\omega)$.

($1 \Rightarrow 2$) Suppose first that RGA fails. Then, for some $\omega \in \Sigma$ and $i \in N$, we have $\hat{\omega} \in \Sigma^{i,\omega}$ such that $\Pi_i(\hat{\omega}) \subseteq S(\hat{\omega})$, but $\hat{\omega} \notin \Pi_i(\hat{\omega})$. Then, $\hat{\omega} \in \Sigma^{i,\hat{\omega}}$, so $\hat{\omega} \in \{\omega' \in \Sigma^{i,\hat{\omega}} : \Pi_i(\omega') = \Pi_i(\hat{\omega})\}$. Hence, $\Pi_i(\hat{\omega}) \neq \{\omega' \in \Sigma^{i,\hat{\omega}} : \Pi_i(\omega') = \Pi_i(\hat{\omega})\}$, that is, $\mathcal{M}$ violates SMIC.

Finally, suppose that ST fails, i.e., there are $\omega, \hat{\omega} \in \Sigma$ such that $\hat{\omega} \in \Pi_i(\omega)$ but $\Pi_i(\hat{\omega}) \neq \Pi_i(\omega)$. Then $\hat{\omega} \notin \{\omega' \in \Sigma^{i,\omega} : \Pi_i(\omega') = \Pi_i(\omega)\}$. Hence, $\mathcal{M}$ violates SMIC. □

Since our current model of the motivating example violates RGA, it is not satisfactory if we require informational consistency. Since RGA is weaker than GR, it might seem there is still a gap that potentially could be exploited to model false beliefs about awareness. However, the next proposition shows that in the presence of C and PPK the gap disappears, as requiring RGA will entail that GR holds as well.

Proposition 4.3 (RGA and GR). *Fix a model $\mathcal{M}$ satisfying C and PPK. The following two statements are equivalent:*

1. *The model $\mathcal{M}$ satisfies RGA.*
2. *The model $\mathcal{M}$ satisfies GR.*

Proof (2 ⇒ 1) is immediate from the definitions of RGA and GR. We prove (1 ⇒ 2). Suppose $\omega \in \Sigma$. If $\Pi_i(\omega) \subseteq S(\omega)$, then by RGA, we have $\omega \in \Pi_i(\omega)$. Since $\Pi_i(\omega) \subseteq \Pi_i(\omega)^{\uparrow}$, GR holds. If, alternatively, $\Pi_i(\omega) \nsubseteq S(\omega)$, then by C, $\Pi_i(\omega) \subseteq S$ for some $S \prec S(\omega)$. Observe that we have $S \preceq S \preceq S(\omega)$, $\omega \in S(\omega)$ and $\Pi_i(\omega) \subseteq S$. Hence by PPK, $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$. Since $(\Pi_i(\omega))_S \subseteq S$, we have $\Pi_i(\omega_S) \subseteq S$. Applying RGA to ω_S, we find $\omega_S \in \Pi_i(\omega_S)$. As observed already, $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$, so $\omega_S \in (\Pi_i(\omega))_S$. But, since $\Pi_i(\omega) \subseteq S$, $(\Pi_i(\omega))_S = \Pi_i(\omega)$, so $\omega_S \in \Pi_i(\omega)$. By Lemma 2.2 (a) (only-if part) we have that $\omega \in \Pi_i(\omega)^{\uparrow}$, that is GR holds. □

In sum, if we want a model that satisfies C, SMIC, and KE, then we cannot accommodate false beliefs. We state this result as a theorem which follows immediately from our previous results.

Theorem 4.4 (Impossiblity of False Beliefs). *There is no model $\mathcal{M}$ that satisfies C, FB, KE and SMIC.*

As a final result, we prove that C, KE and SMIC together imply the full set of HMS conditions.

Theorem 4.5 (KE and SMIC imply HMS). *Fix a model $\mathcal{M}$ satisfying C. The following two statements are equivalent:*

1. *The model $\mathcal{M}$ satisfies KE and SMIC.*
2. *The model $\mathcal{M}$ satisfies the full set of HMS conditions, that is, GR, ST, PPA, PPI and PPK all hold.*

Proof. (1 ⇒ 2): PPK and NIA follow from KE and Proposition 2.4. Then, PPA follows from PPK and C as noted in remark 3 of HMS, p83. Next, GR and ST follow from SMIC and Propositions 4.2 and 4.3. It remains only to show PPI.

Let $\omega \in S''$ and $S \preceq S''$. We need to show that $\Pi_i(\omega)^{\uparrow} \subseteq \Pi_i(\omega_S)^{\uparrow}$. By C, $\Pi_i(\omega) \subseteq S^*$ for some $S^* \preceq S''$, and $\Pi_i(\omega_S) \subseteq S'$ for some $S' \preceq S$. By NIA, $S' \preceq S^*$. We will show the following two things:

1. $\Pi_i(\omega)^{\uparrow} \subseteq \Pi_i(\omega_{S'})^{\uparrow}$;
2. $\Pi_i(\omega_{S'}) = \Pi_i(\omega_S)$.

First, let's see (1). From our results above, we have $S' \preceq S^* \preceq S''$, $\omega \in S''$, and $\Pi_i(\omega) \subseteq S^*$. Hence, $\Pi_i(\omega)^{\uparrow} \subseteq (\Pi_i(\omega))^{\uparrow}_{S'}$. It follows by PPK′ that $(\Pi_i(\omega))_{S'} = \Pi_i(\omega_{S'})$, whence $\Pi_i(\omega)^{\uparrow} \subseteq \Pi_i(\omega_{S'})^{\uparrow}$ as required.

Next, let's see (2). Since $\Pi_i(\omega_S) \subseteq S'$, it follows by GR that $\omega_{S'} \in \Pi_i(\omega_S)$. Hence, by ST, $\Pi_i(\omega_{S'}) = \Pi_i(\omega_S)$ and (2) has been proved. It follows immediately from (1) and (2) that $\Pi_i(\omega)^{\uparrow} \subseteq \Pi_i(\omega_S)^{\uparrow}$, that is, we have proved PPI.

($2 \Rightarrow 1$): As noted by HMS in their remark 2 on p83, PPI and C imply NIA. Hence, KE follows from PPK and NIA using Proposition 2.4. SMIC follows from Propositions 4.2 and 4.3 using GR, ST, and PPK. □

In the presence of C, the requirements of SMIC and KE bring us back to the full HMS conditions. If we want to handle false beliefs, we will have to either violate KE or SMIC. Galanis (2013) chose to violate KE but at the cost of having to redefine events so that knowledge would still be an "event". An alternative way to allow for false beliefs is to violate SMIC as we did in the motivating example.

5. Conclusions

We have shown that as long as the analyst's model $\mathcal{M}$ satisfies the confinement (C) property we can maintain the tradition of each agent reasoning within a sub-model of which they are fully cognizant even in the face of unawareness. Furthermore, this sub-model inherits all the structural properties of $\mathcal{M}$. We can presume that while the game theorist or outside analyst is cognizant of the full model $\mathcal{M}$, each agent i need only be cognizant of his sub-model $\mathcal{M}^{i,\omega}$. The events in the full model $\mathcal{M}$ of which agent i is aware at state ω map neatly into the events in his sub-model $\mathcal{M}^{i,\omega}$.

Moreover, we demonstrated how the generalized reflexivity (GR) property may be dropped and the 'projections preserve information' (PPI) property weakened, while still retaining a number of desirable consistency properties between the events in the original model and the sub-model. This suggests a way to model false beliefs about awareness of others in a suitably modified HMS framework. We provided one motivating example of such a false belief and showed how it could formally be modeled in a suitably modified HMS framework. If, in addition, however, we require the modified HMS framework to satisfy an informational consistency condition, then false beliefs are precluded, and we return to the full set of HMS conditions.

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