

# SAVAGE GAMES

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ABSTRACT. We define and discuss Savage games, which are ordinal games of incomplete information set in L. J. Savage’s framework of purely subjective uncertainty. Every Bayesian game is ordinally equivalent to a Savage game. However, Savage games are free of priors, probabilities and payoffs. Players’ information and subjective attitudes toward uncertainty are encoded in the state-dependent preferences over state contingent action profiles. In the class of games we consider, player preferences satisfy versions of Savage’s *sure thing principle* and *small event continuity* postulate. Savage games provide a tractable framework for studying attitudes towards uncertainty in a strategic setting. The work eschews any notion of objective randomization, convexity, monotonicity, or independence of beliefs. We provide a number of examples illustrating the usefulness of the framework, including novel results for a purely ordinal matching game that satisfies all of our assumptions and for games for which the preferences of the players admit representations from a wide class of decision-theoretic models.

KEYWORDS: subjective uncertainty, strategic interaction, shy events, ambiguity, Bayesian games.

## 1. INTRODUCTION

In his celebrated *Management Science* articles, Harsanyi (1967–1968) introduced the class of Bayesian games to analyze games of incomplete information. These are strategic interactions in which one or more of the participants is uncertain about some relevant aspect of the game being played. As Harsanyi noted, participants might not know precisely what consequence will result from the actions chosen, or they may be unsure about what actions are actually available to the other participants or they may be uncertain about the other participants’ preferences. His key insight was to assume that each participant deals with her incomplete information by assigning a *subjective* probability distribution over all the variables not known to her and chooses an action out of the set of those available to her that maximizes the mathematical expectation of her payoff in terms of that probability distribution. In such a formulation, however, the preferences of the participants rather than being primitives of the model are instead *constructed* from the (cardinal) subjective probabilities and payoffs. Moreover, this reduced-form representation rules out behavior that does not conform to (subjective) expected utility theory, such as, for example, the extensions of expected utility that allow for non-neutral attitudes toward ambiguity.

In this paper, we suggest an alternative way to model the strategic interaction of a finite number of players in the presence of purely subjective uncertainty. To do so, we adopt the framework of Savage’s theory of decision-making under uncertainty, and adapt it to allow for strategic interaction

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among the players. In the process, we introduce a new class of games, denoted *Savage games*, in which the players' objects of choice (strategies) are identified with Savage acts (state-contingent plans), and propose a solution concept that captures the idea of players lacking a strict incentive to deviate at equilibrium. The following desiderata guide our design of a choice-based theory of strategic interaction under subjective uncertainty:

- (1) *Expressed in terms of preferences.* Equilibrium behavior seems more basic than any particular functional form representing utilities. Therefore, we follow the decision-theoretic tradition, and write our assumptions in terms of basic preferences.
- (2) *Consistency with Bayesian games.* The class of games should include Bayesian games as a proper subclass. Moreover, within that subclass the solution concept should be equivalent to Bayesian equilibrium.
- (3) *Consistency with the Savage framework.* In the single-player setting the class of games should be equivalent to Savage's framework for individual decision-making under uncertainty. In particular, it should be able to incorporate generalizations of subjective expected utility such as those allowing for non-neutral attitudes toward ambiguity, thereby allowing for the incorporation of such attitudes in multi-player settings.
- (4) *Theoretical consistency.* The framework should be sufficiently rich to accommodate assumptions that guarantee the existence of equilibrium. These assumptions should be closely related to the assumptions made in models of subjective expected utility and its generalizations. And, ideally, motivated by these assumptions, they should afford us novel insights about the equilibrium of games with expected as well as non-expected utility.
- (5) *Parsimony.* We want the framework and the assumptions imposed therein to be based on the smallest set of elements needed to specify the strategic interaction in which the participants facing subjective uncertainty are engaged.

With all of this in mind, our purpose in this paper is to consider a wider circle of issues than simply the characterization of a class of preferences that admit a particular form of representation. Similarly, our purpose is not to provide an epistemic foundation for decision-making in the presence of subjective strategic uncertainty.

Savage's framework provides a natural starting point for the study of strategic interactions among players in the presence of subjective uncertainty. It allows us, as well as behoves us, to model equilibrium behavior without the usual technical paraphernalia of convexity or monotonicity of strategies and preferences, and the related praxis that seems to have arisen more from considerations of analytical tractability rather than motivated by, for example, behavioral properties of the underlying preferences.

Working with such a framework, however, poses a major challenge as it precludes the use of most of the techniques available in the extant literature on equilibrium theory. First, in a setting that does not involve any notion of objective randomization, the lack of convexity rules out classical calculus approaches as well as the geometric analysis developed over the past century in economic theory. Secondly, without a natural order structure and corresponding intrinsic notion of monotonicity, it is not possible to use the more recent order-theoretic ideas and associated results. Without delving into the technical details, let us just note at this juncture, that our theorem for the presence of an equilibrium is crafted around the standard consequentialist reasoning embodied in Savage's *sure-thing principle* exploiting the ability to move from one best response to another by means of the decomposable choice property of the preferences.

The paper is organized as follows. First, by means of a simple example, we illustrate in section 2 the environment and the main ideas in the model. In Section 3 we describe and study Savage games,

introducing and motivating the assumptions that underpin our main theorem on the existence of equilibrium. In Section 4 we provide an example of a purely ordinal matching game that satisfies all the assumptions required for our existence result, thereby illustrating the notion of ordinal equilibrium. In Section 5 we study games with *recursive* payoffs. We highlight in this section how our assumptions and the result on the existence of equilibrium translate to Bayesian games, games with multiple priors and games in which preferences display other forms of non-expected utility. Section 6 contains two examples with recursive payoffs, the first a Bayesian game and the second with (recursive) multiple priors. We conclude in Section 7 with a discussion of related work, open questions and possible extensions of this work. All the proofs can be found in Appendix A.

## 2. THE ENVIRONMENT AND AN EXAMPLE

Our aim in this paper is to develop an ordinal class of games for analyzing strategic interaction involving incomplete information with purely subjective uncertainty that eschews any *a priori* specification of the cardinal aspects of Bayesian games such as utilities and/or probabilities. Instead any intrinsic attitudes toward uncertainty will be identified (to the extent that this is possible) solely from the (ordinal) preferences of the participants that guide the choices they make in the “play” of the game. Thus everything that any of the participants may be uncertain about is encoded in a *state space* that comprises a collection of mutually exclusive and exhaustive *states* (of the world). As in Savage (1954), each state will be taken to be “a description of the world, leaving no relevant aspect undescribed” (Savage, 1954, p. 9). Although Savage was providing a foundation for a theory of individual choice under uncertainty, he viewed his approach as one that could accommodate considerable generality in terms of the degree of specificity and comprehensiveness of the description of a state of the world. Indeed one of his (motivating) examples of what such a description could conceivably entail was “. . . [t]he exact and entire past, present, and future history of the universe, understood in any sense, however wide” (Savage, 1954, p. 8). For any particular strategic interaction, however, we concur with Savage’s advocacy for “the use of modest little worlds, tailored to particular contexts. . .” (Savage, 1954, p. 9). Thus the formal specification of the state space need only be as rich as is required to accommodate the universe of things that the participants are not sure about and that are *relevant* to the specific strategic interaction being analyzed.

We define a strategy for a player to be a *state-contingent* choice of action. We refer to the collection of strategies available to that player as her strategy set. A consequence will then be anything that may happen as a result of the actions chosen by the participants and the state of the world that obtains. One significant feature of strategic interaction under uncertainty which is not present in Savage’s framework for individual choice under uncertainty is that, in general, different participants will have different information about which state of the world obtains. We shall interpret the information available to a player as corresponding to those subsets of the state-space over which she can “condition” her choice of strategy. That is, we shall interpret any subset of the state space as an (*information*) *event* for this player if she is able to “deviate” from any of her available strategies on that event to any other of her available strategies.

To illustrate these ideas, consider the following strategically interactive elaboration of Savage’s omelet example. Leonard has just broken five good eggs into a bowl when the front doorbell to his apartment sounds. Jimmie his room-mate comes into the kitchen and volunteers to finish making the omelet and to clean up, allowing Leonard to go and open the door which is not visible from the kitchen and is far enough away from the kitchen that anything said there cannot be heard by anyone in the kitchen. They both think the caller could be either their landlord or their neighbor, Jane, on whom, everybody knows, Leonard has a crush. Moreover, everyone knows that the

feelings he has toward Jane are reciprocated by her. Leonard is unsure, however, about Jimmie's attitude toward this budding romance. In particular, he does not know whether Jimmie is envious or happy for them. If the caller turns out to be Jane, then Leonard must decide whether or not to invite her to come in to share their omelet.<sup>1</sup> A sixth egg, intended for the omelet, lies unbroken beside the bowl containing the five good eggs. Jimmie must decide either to break this egg into the bowl containing the five good eggs or to break it into a saucer affording him the opportunity to check whether it is rotten before adding it to the other five good eggs.

For some reason Leonard and Jimmie (must) make their respective decisions simultaneously. To ensure that it is only Leonard and Jimmie who have a non-trivial decision to make, let us assume that Jane will accept for sure any invitation to come in and share their omelet.

In this description of strategic interaction under uncertainty in which Leonard and Jimmie are the two players, there are three things that either one or both of them are uncertain about:

- (1) whether or not the egg is rotten;
- (2) whether the caller at the door is their landlord or is Jane;
- (3) whether Jimmie is envious of or is happy about Leonard and Jane's romance.

As these three things are at least logically distinct, it is natural to model the uncertainty with a state space that has  $2^3$  elements. For example, we could specify the state space as follows:

$$\Omega = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$$

where each  $\omega = rst \in \Omega$  is the state in which

$$r = \begin{cases} 1 & \text{if the sixth egg is rotten,} \\ 0 & \text{if the sixth egg is good.} \end{cases}$$

$$s = \begin{cases} 1 & \text{if the caller at the door is the landlord,} \\ 0 & \text{if the caller at the door is Jane.} \end{cases}$$

$$t = \begin{cases} 1 & \text{if Jimmie is envious of Leonard and Jane's romance,} \\ 0 & \text{if Jimmie is happy about Leonard and Jane's romance.} \end{cases}$$

From the description above we can also identify four distinct actions that may be taken in the course of "play". So, for example, we could take the action set  $A$  to be the four-element set  $\{a_1, a_2, a_3, a_4\}$ , where action

- $a_1$  is breaking the sixth egg into the bowl containing the other five good eggs.
- $a_2$  is breaking the sixth egg into a saucer for inspection.
- $a_3$  is inviting the caller at the door in to share the omelet.
- $a_4$  is not inviting the caller at the door in to share the omelet.

A consequence can then be associated with each (feasible) action pair and state combination. For example, the action pair and state combination  $(a_1, a_4, 111)$  is associated with the consequence that results when Jimmy who is envious of Leonard and Jane's romance, breaks a rotten egg into the bowl containing the other five good eggs. This saves Jimmy the bother of having to wash an extra saucer, but in this instance at the cost of ruining the omelet. Moreover, when Leonard opens the door, he finds it is their landlord, to whom he does not extend an invitation to come in to join them and share their omelet.

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<sup>1</sup>We suppose that their relationship with their landlord although proper and polite is not one that could be said to be "familiar" to any meaningful degree and so neither would ever consider it appropriate to invite their landlord in to share a meal.

We take the strategies available to our players, Leonard and Jimmie, to be those mappings from the state space  $\Omega$  to the action set  $A$  that are *consonant* with the above description of their strategic interaction under uncertainty in terms of what each player knows and does not know and consequently what actions are available to him. For example, if  $f_J$  denotes a strategy available to Jimmie, then its range must be a non-empty subset of  $\{a_1, a_2\}$ , since Jimmie does not go to answer the door and so cannot invite Jane to share the omelet, should she be the one who was ringing their front doorbell. Moreover, assuming that Jimmie knows his own feelings about Leonard and Jane's romance, then the only aspect of the state  $\omega = rst$  on which he can condition his decision whether to select action  $a_1$  or action  $a_2$  is the " $t$ ". Notice, when he is deciding between breaking the sixth egg into the bowl with the five good eggs or breaking it into a separate saucer, he does not know whether the egg is rotten or good, nor does he know who is the caller at the door.

If we denote the set of strategies available to Jimmie by  $F_J$ , it follows from the above description of the strategic interaction that

$$F_J = \{f_J: \Omega \rightarrow \{a_1, a_2\} \mid f(00t) = f(01t) = f(10t) = f(11t), t = 1, 2\}.$$

Correspondingly, if  $f_L$  denotes an available strategy for Leonard, then it follows from the description of the situation above that, for any state  $\omega = rst$  with  $s = 1$  (that is, the event in which the caller at the door is the landlord),  $f_K(\omega) = a_4$ , since we have assumed, given the nature of their relationship with their landlord they would never contemplate inviting him in to share a meal.<sup>2</sup> So, if we denote the set of strategies available to Leonard by  $F_L$ , then it follows from our story above that

$$F_L = \{f_L: \Omega \rightarrow \{a_3, a_4\} \mid f(111) = a_4 \text{ and } f(0s0) = f(0s1) = f(1s0) = f(1s1), s = 1, 2\}.$$

Notice that we can associate with each strategy profile  $(f_J, f_L)$  in  $F_J \times F_L$  the following mapping from states to consequences

$$\omega \mapsto (f_J(\omega), f_L(\omega), \omega)$$

Thus, from the perspective of the analyst or modeler, the *ex ante* uncertainty facing the players given they collectively choose their actions according to that strategy profile is embodied in this associated *state-contingent consequence*. It is therefore natural to assume that each player's choice of strategy will be guided by her underlying preferences over state-contingent consequences that will be reflected by a binary relation defined over the set of strategy profiles. Hence, to complete the specification of the ordinal normal-form game for modeling Leonard and Jimmie's strategic interaction under uncertainty, requires two binary relations  $\succsim_J, \succsim_L \subset F_J \times F_L$  corresponding to the (*ex ante*) preferences of Jimmie and Leonard, respectively, that will guide their strategy choice.

An equilibrium for this ordinal normal-form game is a strategy profile  $(f_J^*, f_L^*)$  for which

$$\begin{aligned} (f_J^*, f_L^*) &\succsim_J (f_J, f_L^*), \text{ for all } f_J \in F_J \\ \text{and } (f_J^*, f_L^*) &\succsim_L (f_J^*, f_L), \text{ for all } f_L \in F_L \end{aligned}$$

Notice that for the purpose of finding such an equilibrium (should one exist) there is no need for one player to be able to express a preference between two strategy profiles that involve *different*

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<sup>2</sup>Alternatively we could consider expanding Leonard's strategy set to include strategies that involve him extending an invitation to the landlord and through the specification of his preferences ensure that it would never be a best response for him to select any strategy that involved him inviting the landlord in. However, unless we presumed a pre-determined response by the landlord this would mean the landlord would become a third strategic player whose behavior we would need to consider. Moreover, our purpose here is to illustrate how the uncertainty one player may have about the actions available to another player can be modeled within the Savage game framework.

strategy choices by the other player. Although such a preference may be plausible from the description of the strategic interaction, we note that it involves comparisons across pairs of strategy profiles that cannot inform us about how that player might choose her strategy in the ordinal game. For example, for any given strategy of Leonard's  $f_L \in F_L$ , if Leonard had the following strict preference  $(f_J, f_L) \succ_L (f'_J, f_L)$ , (where  $\succ_L$  denotes the asymmetric component of  $\succsim_L$ ) then no matter what preferences Jimmie had, this strict preference could never guide Leonard's choice of strategy since to move from the latter strategy profile to the former can only be achieved by a change in strategy choice by Jimmie. Although Leonard might strictly prefer the former to the latter, which of these two strategy profiles might be played should Leonard choose the strategy  $f_L$  is a choice that would be made by Jimmie and so presumably would be guided by his preferences, *not* Leonard's. We simply note at this juncture that the assumptions we impose on the preferences of players in order to establish the existence of an equilibrium of the ordinal game do not require any player to be able to express a preference between any pair of strategy profiles that involves a different strategy choice by any player other than himself.

### 3. SAVAGE GAMES

Following the discussion of the previous section, we now present the formal description of a *Savage game*. It is specified by the ternion

$$(\Omega, A, (F_i, \succsim_i)_{i=1}^N).$$

The set  $\Omega$  denotes the common *state space* and  $A$  is the common nonempty *action set*, which we take to be a compact metric space. The tuple  $(F_i, \succsim_i)_{i=1}^N$  is an  $N$ -player ordinal game whose parameters are described below.

There are  $N \geq 1$  players indexed by  $i = 1, \dots, N$ . We abuse notation by having  $N$  also denote the set  $\{1, \dots, N\}$ . However, we employ standard notation for the indexing of player profiles. In particular, for any  $N$ -tuple  $(Z_i)_{i=1}^N$  of sets we write  $Z$  for its  $N$ -ary Cartesian product and for each player  $i$  we write  $Z_{-i}$  for the Cartesian product of the tuple  $(Z_j)_{j \neq i}$ . Vectors in  $Z$  are called *profiles* and vectors in  $Z_{-i}$  are called *profiles of players other than  $i$* . A profile  $z \in Z$  is also written as  $(z_i, z_{-i})$  where  $z_i$  is the  $i$ -th coordinate of  $z$  and  $z_{-i}$  is the projection of  $z$  into  $Z_{-i}$ .

Player  $i$  has a non-empty set  $F_i$  of  $A$ -valued functions on the state space  $\Omega$  called the *strategy space*. A function  $f_i: \Omega \rightarrow A$  in  $F_i$  is called a *strategy* for player  $i$ . Let  $F$  be the set of strategy profiles and for each  $i$  let  $F_{-i}$  be the set of strategy profiles of players other than  $i$ .

Player  $i$  is also associated with a binary relation  $\succsim_i$  on the set of strategy profiles  $F$  describing her weak preferences. Let  $\sim_i$  represent the indifference relation associated with  $\succsim_i$ , that is  $f \sim_i g$  if  $f \succsim_i g$  and  $g \succsim_i f$ . Our first assumption, A1, requires the preferences of the players to be complete and transitive with respect to their own choices.

**A1.** For any  $f \in F$  and any  $g_i, h_i \in F_i$ ,

- (1) either  $(f_i, f_{-i}) \succsim_i (g_i, f_{-i})$  or  $(g_i, f_{-i}) \succsim_i (f_i, f_{-i})$ , and
- (2) if  $(f_i, f_{-i}) \succsim_i (g_i, f_{-i})$  and  $(g_i, f_{-i}) \succsim_i (h_i, f_{-i})$  then  $(f_i, f_{-i}) \succsim_i (h_i, f_{-i})$ .

A strategy profile  $f \in F$  is an *equilibrium* if

$$f \succsim_i (g_i, f_{-i})$$

for all  $g_i \in F_i$  and  $i \in N$ .

In a Savage game the information available to a player is encoded in the specification of the set of strategies  $F_i$ . Following standard notation, for any subset  $E \subseteq \Omega$  and two functions  $f_i, g_i: \Omega \rightarrow A$

let  $g_{iE}f_i$  be the function from  $\Omega$  to  $A$  given by

$$g_{iE}f_i(\omega) = \begin{cases} g_i(\omega) & \text{if } \omega \in E, \\ f_i(\omega) & \text{otherwise.} \end{cases}$$

We refer to the function  $g_{iE}f_i$  as the  $g_i$ -deviation from  $f_i$  conditional on  $E$ .

**Information events.** A set of states  $E \subseteq \Omega$  is an (information) *event* for Player  $i$  if she can condition her choice of strategy on  $E$ , that is,  $g_{iE}f_i \in F_i$  for all  $f_i, g_i \in F_i$ . Denote by  $\mathcal{F}_i$  the family of events for Player  $i$ .

One way to interpret the information structure of a player is to view the Savage game as a dynamic game. In the first stage, each Player  $i$  receives partial information about the true state of the world, encoded in  $\mathcal{F}_i$ , the set of information events for Player  $i$ . Only after that, in the *interim* stage, do players make their choice of action. Then in the last stage, all uncertainty is resolved and the final consequence is realized.

With this interpretation of the information structures of the players in mind, the next assumption can be viewed as being motivated by the standard consequentialist reasoning embodied in Savage's *sure thing principle*. It is, in fact, based on the normative rule *dynamic programming solvability* introduced by Gul and Lantto (1990) in the context of individual choice under uncertainty. Gul and Lantto highlight that although dynamic programming solvability constitutes a weakening of Savage's postulate **P2**, it still allows for a simplification of the task faced by an individual in finding an optimal "plan of action" for a two-stage decision tree by allowing the player to "fold back" or "roll back" the two-stage decision tree. To illustrate the idea, Gul and Lantto give the example of a commuter having to decide how to go to work. The options are to walk, to drive, to bike or to take the bus. Suppose the following two plans are optimal: (1) drive if it rains, bike if it is sunny; and (2) take the bus if it rains, walk if it is sunny. Then, they argue, the following plans of actions should also be optimal: (3) drive if it rains, walk if it is sunny; and (4) take the bus if it rains, bike if it is sunny. In the setting of a Savage game this translates into the requirement that piecewise combinations of best responses (consistent with that player's information) remain a best response.

In what follows, we shall omit the quantifiers from our assumptions when they are obvious. In particular,  $f$  is understood as an arbitrary member of  $F$ ,  $f_i$  and  $g_i$  of  $F_i$ , and  $E$  always denotes an event in  $\mathcal{F}_i$ .

**A2.** If  $(f_i, f_{-i}) \sim_i (g_i, f_{-i}) \succsim_i (h_i, f_{-i})$  for all  $h_i \in F_i$ , then  $(g_{iE}f_i, f_{-i}) \sim_i (f_i, f_{-i})$  for all  $E \in \mathcal{F}_i$ .

The reader will see in Section 5 that in games in which the preferences are given by payoffs, A2 holds when a wide range of assumptions that have been studied in the literature are satisfied. In particular and as has been already foreshadowed above, the following proposition establishes A2 is implied by Savage's postulate **P2**.

**Proposition 3.1.** *The following condition implies A2:*

**P2:** *If  $(f_i, f_{-i}) \succsim_i (g_{iE}f_i, f_{-i})$ , then  $(f_i E g_i, f_{-i}) \succsim_i (g_i, f_{-i})$ .*

We do not assume that the game contains constant strategies or that it is non-degenerate in the sense of Savage. We make, however, the following "richness" assumption on strategies. It basically states that every state-contingent plan of action that a player can approximate by sequences of strategies must also be a feasible strategy for that player.

**A3.** If  $E^n \in \mathcal{F}_i$  is an increasing sequence of events for Player  $i$ , then  $g_{i(\cup_n E^n)}f_i \in F_i$  for every  $f_i, g_i \in F_i$ .

Notice that, by construction,  $\mathcal{F}_i$  is a collection of subsets of  $\Omega$  (events) that contains  $\Omega$ , and is closed under the operations of complement and finite unions, that is,  $\mathcal{F}_i$  is an algebra of sets. The next proposition establishes that, if Assumption A3 holds, then  $\mathcal{F}_i$  is also closed under countable unions, that is, it is a  $\sigma$ -algebra.

**Proposition 3.2.** *The collection of events for Player  $i$ ,  $\mathcal{F}_i$ , is an algebra over  $\Omega$ . If A3 holds, then  $\mathcal{F}_i$  is a  $\sigma$ -algebra.*

When the structure of a Bayesian game is available, we have the following corollary, that shows that  $\mathcal{F}_i$  is precisely the smallest  $\sigma$ -algebra for which all Player  $i$ 's strategies, as usually defined, are measurable.

**Corollary 3.3.** *Let  $\Sigma_i$  be a  $\sigma$ -algebra over  $\Omega$  and  $A$  be a compact metric space with  $|A| \geq 2$ . If  $F_i$  is the set of all  $\Sigma_i$ -measurable functions to  $A$ , then A3 holds and  $\mathcal{F}_i = \Sigma_i$ .*

We extend the concept of a Savage null event to our setting of interdependent preferences. To do so, first, it is useful to define what it means for a pair of strategy profiles to be *strategically equivalent* for a player.

**Strategic equivalence.** The strategy profiles  $f, g \in F$  are *strategically equivalent* for Player  $i$ , denoted  $f \approx_i g$ , if for all  $h_i \in F_i$ ,

$$(h_i, f_{-i}) \succsim_i (f_i, f_{-i}) \iff (h_i, g_{-i}) \succsim_i (g_i, g_{-i})$$

$$\text{and } (f_i, f_{-i}) \succsim_i (h_i, f_{-i}) \iff (g_i, g_{-i}) \succsim_i (h_i, g_{-i}).$$

That is, if  $f \approx_i g$ , then Player  $i$ 's preferences over her own strategies in  $F_i$  are the same when the other players are choosing the profile  $f_{-i}$  as when they are choosing the profile  $g_{-i}$ .

The following properties readily follow from the definition of strategic equivalence.

**Corollary 3.4.** *The binary relation  $\approx_i$  is an equivalence relation (that is, it is reflexive, symmetric and transitive).*

**Corollary 3.5.** *For any  $f \in F$  and any  $g_i \in F_i$ ,  $(f_i, f_{-i}) \approx_i (g_i, f_{-i})$  if and only if  $(f_i, f_{-i}) \sim_i (g_i, f_{-i})$ .*

We shall refer to the strategic analog of a null event as a *strategically shy* event. An event will be deemed strategically shy for a player if any deviation that player can make conditional on that event from any strategy profile leaves that player indifferent *and* does not affect any *other* player's preferences over her own strategies. That is, the original strategy profile and the strategy profile resulting from that player's deviation are *strategically equivalent for every player*.

**Strategically shy events.** An event  $E \in \mathcal{F}_i$  is *strategically shy* for Player  $i$  if for all  $f \in F$  and all  $g_i \in F_i$  we have  $(g_i E f_i, f_{-i}) \approx_j (f_i, f_{-i})$ , for every player  $j \in N$ . Denote by  $\mathcal{N}_i$  the set of all events that are shy for Player  $i$ . Let  $\mathcal{R}_i = \mathcal{F}_i \setminus \mathcal{N}_i$  be the set of *strategically relevant* events for Player  $i$ .

Notice that two players  $i, j \in N$  may share an event  $E \in \mathcal{F}_i \cap \mathcal{F}_j$  that is strategically shy for Player  $i$  but relevant for Player  $j$ . We do not view this as anomalous or inconsistent. It simply means that when conditioning on this event, no deviation by Player  $i$  has any strategic relevance for any of the players. However, there exists at least one deviation by Player  $j$  that is strategically relevant either for that player or for at least one of the other players. Even in the context of individual choice under uncertainty, Karni, Schmeidler, and Vind (1983) note that if preferences are state-dependent,



then interpreting null events as ones that are necessarily viewed by the decision-maker as having zero probability of occurring is problematic. For example, if one of the events involves loss of life then its nullity could reflect the decision-maker having no strict preference about which outcome obtains in the event she is dead, rather than her believing she has no chance of dying.

Turning to continuity, we require the preferences of the players to be continuous with respect to state-wise converging sequences of strategy profiles. Notice that condition A4 is satisfied in continuous Bayesian games in which the payoff of the players is computed by means of an integral.

**A4.** If  $f_i^n \in F_i$ ,  $g_i^n \in F_i$ , and  $f_{-i}^n \in F_{-i}$  are sequences converging state-wise to  $f_i \in F_i$ ,  $g_i \in F_i$ , and  $f_{-i} \in F_{-i}$ , respectively, and  $(g_i^n, f_{-i}^n) \succsim_i (f_i^n, f_{-i}^n)$  for all  $n$ , then  $(g_i, f_{-i}) \succsim_i (f_i, f_{-i})$ .

By construction, the set of strategically shy events for Player  $i$  is closed with respect to subsets and finite unions, that is, it is an ideal. The next proposition establishes that, if assumption A4 holds, then  $\mathcal{N}_i$  is also closed under countable unions, that is, it is a  $\sigma$ -ideal.

**Proposition 3.6.** *The collection of shy events for Player  $i$ ,  $\mathcal{N}_i$ , is an ideal in  $\mathcal{F}_i$ . If A3 and A4 hold for all players, then  $\mathcal{N}_i$  is a  $\sigma$ -ideal.*

The next assumption is a “fullness” assumption on strategically relevant events. A family of events  $\mathcal{S} \subseteq \mathcal{F}_i$  is *closed* if for any increasing sequence of events  $E^n$  in  $\mathcal{S}$  whose union  $E$  is an event, we have  $E \in \mathcal{S}$ .

**A5.** There is a sequence of closed families  $\mathcal{S}_i^m$  of events satisfying:

- (1) If  $E \in \bigcap_m \mathcal{S}_i^m$ , then  $E$  is strategically shy for Player  $i$ .
- (2) If  $E^n$  is a sequence of strategically relevant events for Player  $i$  and

$$\liminf_{n \rightarrow \infty} \max_{\omega \in \Omega} \frac{1}{n} |\{1 \leq k \leq n : \omega \in E^k\}| = 0,$$

then for each  $m$  there is  $n$  such that  $E^n \in \mathcal{S}_i^m$ .

Assumption A5 can be interpreted as follows. Condition (1) indicates that the sequence  $\mathcal{S}_i^m$  comprises families of small events forming a neighborhood base for the subfamily of shy events. The expression  $\frac{1}{n} |\{1 \leq k \leq n : \omega \in E^k\}|$  is the average incidence of state  $\omega$  arising from the sequence of events  $E^n$ . Keeping that in mind, condition (2) then implies that players understand shy events to be limits of decreasing sequences of relevant events.

Assumption A5 adapts a condition of Ryll-Nardzewski and Kelley that is sufficient for existence of a measure on a  $\sigma$ -algebra (see the addendum to Kelley (1959)). An alternative to assumption A5 would be to follow the approach taken by Arrow (1971), Fishburn (1970), and Villegas (1964), and construct a probability measure over a given  $\sigma$ -algebra based on a qualitative probability.<sup>3</sup> The reason we do not follow that approach is it entails monotonicity assumptions, which in turn imply state-independent preferences. In the context of a Bayesian game, that implies that each player *ex post* payoff cannot depend on the players’ types, thus ruling out an important class of Bayesian games.

The following proposition establishes the equivalence between A5 and existence of a measure on Player  $i$ ’s collection of events  $\mathcal{F}_i$ .

<sup>3</sup>A binary relation  $\geq$  is a qualitative probability if

- (1)  $\geq$  is a weak order (reflexive, complete and transitive);
- (2)  $E \geq \emptyset$  for every event  $E$ ; and
- (3)  $E' \cap E = E'' \cap E = \emptyset$  implies  $[E' \geq E'' \iff E' \cup E \geq E'' \cup E]$ .

**Proposition 3.7.** *If A3, A4 hold for all players, then the following are equivalent:*

- (1) *Assumption A5 holds for Player  $i$ .*
- (2)  *$\mathcal{F}_i$  admits a measure  $\pi_i$  such that  $\pi_i(E) = 0$  if and only if  $E \in \mathcal{N}_i$ .*

As an example, suppose  $\Omega = [0, 1]$ . Suppose also that for a particular Player  $i$  the set  $\mathcal{F}_i$  is the Lebesgue subsets of  $\Omega$ , and  $\mathcal{N}_i$  is the collection of subsets of zero Lebesgue measure,  $\lambda$ . In this case, one possibility is for the family of events  $\mathcal{S}_i^m$  to be given by  $\mathcal{S}_i^m = \{E \in \mathcal{F}_i : \lambda(E) \leq \frac{1}{m}\}$ . Then one example of a sequence of strategically relevant events satisfying A5(2) is given by  $\{E^{nm}\}$ , in which  $E^{nm} = [\frac{m-1}{n}, \frac{m}{n})$ ,  $n \geq 1$ , and  $m = 1, \dots, n$ . In this example, the measure  $\pi_i$  whose existence is guaranteed by Proposition 3.7 could be *any* measure whose collection of zero measure sets coincides with the collection of zero Lebesgue measure sets,  $\mathcal{N}_i$ .

The next assumption is an interdependent version of Savage's postulate **P6**, which is usually interpreted as a *small event continuity* property.

**A6.** If  $(f_i, f_{-i}) \not\approx_j (g_i, f_{-i})$  for some  $j \in N$ , then for each  $h_i \in F_i$  there exist events  $\{E^1, \dots, E^n\}$  such that  $\cup_k E^k = \Omega$  and  $(f_i, f_{-i}) \not\approx_j (h_i E^k g_i, f_{-i})$  for all  $k$ .

As with **P6**, this assumption ensures that each relevant event can be split into two disjoint relevant events. Therefore, A6 implies that the measure  $\pi_i$  over  $\mathcal{F}_i$  from Proposition 3.7 is atomless.

**Proposition 3.8.** *If A6 holds, then every  $E \in \mathcal{R}_i$  contains two disjoint events in  $\mathcal{R}_i$ .*

The following proposition asserts that, in fact, under A3 and A4, assumptions A5 and A6 are equivalent to measure  $\pi_i$  being atomless.

**Proposition 3.9.** *If A3 and A4 hold for all players, then the following are equivalent:*

- (1) *Assumptions A5 and A6 hold for Player  $i$ .*
- (2)  *$\mathcal{F}_i$  admits an atomless measure  $\pi_i$  such that  $\pi_i(E) = 0$  implies  $E \in \mathcal{N}_i$ .*

Notice that in condition (2) some strategically shy events of a player may be given a positive measure. The only requirement is that all events assigned zero measure are strategically shy for that player. Since we are dealing with countably additive measures, however, there exists a corresponding measure that is positive on strategically relevant events and zero on all strategically shy events.

Our final assumption imposes a restriction on the best responses of players that is essential for existence of equilibrium in the Savage game. It plays the role of a compactness assumption in more standard equilibrium existence proofs. To be more specific, assumption A7 requires players to be able to find best responses that are not too erratic.

**Regularity of best responses.** A subset of strategies  $X_i \subseteq F_i$  is said to be *countably distinguished* if there exists a countable set of states  $\mathbb{W} \subseteq \Omega$  such that for any distinct  $f_i, g_i \in X_i$  we have  $f_i(\omega) \neq g_i(\omega)$  for some  $\omega \in \mathbb{W}$ .

**A7.** For each  $i$  there is a set  $X_i \subseteq F_i$  of strategies satisfying the following:

- (1) For each  $f \in F$ , there is  $g_i \in X_i$  satisfying  $(g_i, f_{-i}) \succsim_i (f_i, f_{-i})$ .
- (2)  $X_i$  is countably-distinguished and every sequence in  $X_i$  has a subsequence converging state-wise to a strategy in  $X_i$ .

The significance of A7 can be seen in some well-known examples of Bayesian games that fail to have pure-strategy equilibrium. The following example is taken from Radner and Rosenthal (1982).<sup>4</sup>

<sup>4</sup>Another particularly striking example is presented in Khan, Rath, and Sun (1999). An argument similar to the one presented for Example 3.10 shows that A7 fails to hold.

**Example 3.10.** There are two players, 1 and 2. Each player has two actions, thus  $A_1 = A_2 = \{0, 1\}$ . Payoffs are zero-sum, according to the following payoff matrix:

		Player 2	
		0	1
Player 1	0	(1, -1)	(-1, 1)
	1	(-1, 1)	(1, -1)

The players type space is the unit square  $[0, 1]^2$ . The players' types are distributed uniformly on the triangle of the unit square given by  $\{(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq 1\}$ . Radner and Rosenthal show that this game has no pure-strategy equilibrium. At any equilibrium, for each player  $i$ , conditional on his type being  $t$ , the probability of the other player choosing action 0 has to be  $\frac{1}{2}$ . That is, for a pair of pure strategies  $(f_1, f_2)$  to be an equilibrium in this game, it is necessary and sufficient that

$$\frac{\lambda([t_1, 1] \cap f_2^{-1}(0))}{(1 - t_1)} = \frac{\lambda([0, t_2] \cap f_1^{-1}(0))}{t_2} = \frac{1}{2}$$

for almost every  $t_1$  and almost every  $t_2$ , with  $\lambda$  denoting the Lebesgue measure. Therefore, if it existed, at a pure-strategy equilibrium, the strategies played would be very erratic. In fact, it is possible to construct a sequence of strategies of player 2 to which player 1's (essentially unique) best responses become increasingly erratic, and thus cannot be contained in a subset of strategies of player 1 that satisfies A7. To illustrate that statement, we provide the first few elements of one possible such erratic sequence of strategy profiles. Consider the following sequence of strategies of player 2:

$$\begin{aligned} f_2^1(t_2) &= 0 \\ f_2^2(t_2) &= \begin{cases} 0 & \text{if } t_2 \in [0, 0.5] \\ 1 & \text{if } t_2 \in [0.5, 1]. \end{cases} \\ f_2^3(t_2) &= \begin{cases} 0 & \text{if } t_2 \in [0, 0.3] \cup (0.6, 1] \\ 1 & \text{if } t_2 \in [0.3, 0.6]. \end{cases} \\ &\dots \end{aligned}$$

The unique (up to a zero Lebesgue measure set) best responses of player 1 would then be

$$\begin{aligned} f_1^1(t_1) &= 0 \\ f_1^2(t_1) &= \begin{cases} 0 & \text{if } t_1 \in [0, 0.5\sqrt{2}] \\ 1 & \text{if } t_1 \in [0.5\sqrt{2}, 1]. \end{cases} \\ f_1^3(t_1) &= \begin{cases} 0 & \text{if } t_1 \in [0, 0.3\sqrt{2}] \cup (0.3\sqrt{6}, 1] \\ 1 & \text{if } t_1 \in [0.3\sqrt{2}, 0.3\sqrt{6}]. \end{cases} \\ &\dots \end{aligned}$$

It is possible to continue the sequence of strategy profiles  $\{(f_1^1, f_2^1), (f_1^2, f_2^2), (f_1^3, f_2^3), \dots\}$  such that each element  $(f_1^n, f_2^n)$  has exactly  $n - 1$  discontinuity points. Moreover, because each  $f_1^n$  is the (essentially) unique best response to player 2's strategy  $f_2^n$ , for A7 to be satisfied the entire sequence  $\{f_1^n\}$  must be contained in  $X_1$ . However, this sequence has no convergent subsequence.

Assumption A7(2) is equivalent to saying that  $X_i$  is metrizable and compact in the sequential topology. For clarity, we provide some examples of such spaces.

- (a)  $X_i$  is a finite set.
- (b)  $X_i$  contains a countable set and its countable accumulation points.
- (c) If  $\Omega$  and  $A$  are compact metric spaces, then any closed collection of equicontinuous strategies satisfies this assumption. In particular,  $X_i$  is the set of all Lipschitz continuous functions with common constant.<sup>5</sup>
- (d) Suppose that  $\Omega$  is the set of all continuous functions from  $[0, 1]$  to  $\mathbb{R}$ , and  $A$  is the set of Radon measures on  $[0, 1]$ . Then the set of all Radon probability measures with the weak\*-topology has the required property.<sup>6</sup>
- (e) Suppose that  $\Omega = [0, 1]$  and  $A = \mathbb{R}^d$ . For any compact set of functions  $Y$  in  $L_\infty$ , there is a selection from the equivalence classes of these functions that is compact and metrizable in the sequential topology.<sup>7</sup>
- (f) Suppose that  $\Omega$  is a measure space with  $\sigma$ -algebra  $\mathcal{F}$ , and  $A$  is a Banach space. This assumption is satisfied whenever  $X_i$  is a sequentially compact set of bounded measurable functions for the topology of state-wise convergence, and there is a probability measure  $\pi$  on  $\mathcal{F}$  such that if  $f_i$  and  $g_i$  are distinct functions in  $X_i$ , then they differ on a set of positive  $\pi$  measure.

We are now ready to state the main result.

**Theorem 3.11.** *If A1–A7 hold for all players, then an equilibrium exists.*

The proof of Theorem 3.11 is in Appendix A, but we conclude this section with a brief overview of the relevance of each assumption. We prove the theorem by showing that the product correspondence of the players' best responses has a fixed point. Assumptions A1 and A3–A5 guarantee that each player's best response correspondence is well-defined and has a closed graph.<sup>8</sup> Assumptions A2 and A6 guarantee that the values of each player's best response correspondence is path connected, which plays the role of convexity in standard fixed point proofs.<sup>9</sup> Finally, assumption A7 is used to show that the product correspondence of best responses has a subcorrespondence whose image is contained in a compact subset of the strategy space. Moreover, this subcorrespondence

<sup>5</sup>A family of functions  $F$  between metric spaces is called equicontinuous if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_1(x, y) < \delta$  implies  $d_2(f(x), f(y)) < \varepsilon$  for every  $f \in F$ . A function  $f$  between metric spaces is Lipschitz continuous if there exists a constant  $K$ , called Lipschitz constant, such that  $d_2(f(x), f(y)) < Kd_1(x, y)$ .

<sup>6</sup>A Radon measure  $\mu$  is a measure on the  $\sigma$ -algebra of Borel sets of a Hausdorff topological space  $X$  that is both locally finite (every point of  $X$  has a neighborhood  $U$  for which  $\mu(U)$  is finite) and inner regular (for any Borel subset  $B$  of  $X$ ,  $\mu(B)$  is the supremum of  $\mu(K)$  over all compact subsets  $K$  of  $B$ ). The weak\*-topology on a set  $\mathcal{M}$  of measures on a measurable space  $(\Omega, \mathcal{F})$  is the weakest topology such that all the linear functionals  $L_f: \mu \mapsto \int_\Omega f d\mu$  for  $f: \Omega \rightarrow \mathbb{R}$   $\mathcal{F}$ -measurable and bounded, are continuous.

<sup>7</sup>Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space, and  $A$  a Banach space. Then  $L_\infty(\Omega, \mathcal{F}; \mu)$  is the set of (equivalence classes of) essentially bounded functions from  $\Omega$  to  $A$  endowed with the essential supremum norm:

$$\|f\|_\infty = \inf\{c > 0: |f(\omega)| \leq c \text{ for } \mu\text{-a.e. } \omega\}.$$

<sup>8</sup>A correspondence  $B: X \rightarrow Y$  between topological spaces is said to have closed graph if the set  $\{(x, y) \in X \times Y: y \in B(x)\}$  is a closed subset of  $X \times Y$ .

<sup>9</sup>A topological space  $X$  is path connected if any two given elements of it can be joined by a path, that is, given  $x, y \in X$ , there exists a continuous map  $f: [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ .

inherits the path-connectedness and closed graph properties of the larger best response correspondence. Therefore, this allows us to apply the Eilenberg and Montgomery fixed point theorem to this subcorrespondence.

#### 4. MATCHING WITH INCOMPLETE INFORMATION

To illustrate the main concepts, definitions and structures of the framework developed in the previous section, we present a fully specified example of a purely ordinal matching process that satisfies all of our assumptions, hence establishing the existence of an equilibrium match. This model can be viewed as an abstraction of the academic job market, or the system of academic admissions. As far as we know, there are no other examples in the literature of incomplete information matching games without values, beliefs, and without a specification of a utility representation for the preferences of the players.

There is a finite set of firms,  $N = \{1, \dots, N\}$ ; we denote an individual firm by  $i \in N$ . There is also a finite set of workers,  $W = \{1, \dots, W\}$ ; we denote an individual worker by  $j \in W$ . The end result of the matching process in this job market for worker  $j$  is denoted by a pair  $(i, b)$ . The first element  $i \in N \cup \{0\}$  is the firm to which this worker is allocated, with the interpretation that 0 corresponds to the worker not being employed by any of the firms in the market. The second element  $b$  specifies for each worker  $j$ , the remuneration bundle that firm  $i$  pays worker  $j$ .

To avoid dealing with measurability issues and for ease of exposition, we identify the space  $\mathcal{B}$  of feasible remuneration bundles with a *finite* subset of some Euclidean space  $\mathbb{R}^\ell$ , with  $\ell \geq 1$ . As an example, one dimension could represent the salary offer, towards which workers have strictly monotone preferences. Other dimensions could be the levels of health and retirement benefits. Still other dimensions could represent non-monetary compensations, such as the city in which the job is to be performed. We assume  $0 \in \mathcal{B}$  and take this to be the “default bundle” offered to any worker that is allocated to “firm” 0, as well as the “default bundle” offered by any firm to a worker with whom the firm does not want to be matched.

Hence formally, an outcome of this matching game can be taken to be a mapping

$$\theta : W \mapsto (N \times \mathcal{B}) \cup \{0, 0\},$$

where for each worker  $j$ ,  $\theta_1(j)$  is the firm at which worker  $j$  ends up employed and  $\theta_2(j)$  is her remuneration bundle. Let  $\Theta$  denote the (by construction, finite) set of all possible outcomes of the matching process.

So that workers need not consider any strategic interactions when making their choices, we assume that each worker only cares about the specifics of her own match. That is, worker  $j$  is indifferent between any pair of outcomes  $\theta$  and  $\theta'$  for which  $\theta(j) = \theta'(j)$ . In particular, each worker  $j$  in  $W$  is characterized by a binary relation  $P_j$  defined on  $N \times \mathcal{B} \cup \{0, 0\}$ , that is complete, transitive and (so that best responses will be unique) asymmetric. That is,  $P_j$  is a (complete) strict preference relation. In addition, we suppose

$$(0, 0) P_j (i, 0) \text{ for every } i \in N \text{ and every } j \in W.$$

Asymmetry of  $P_j$  precludes  $(i, 0) P_j (0, 0)$  from holding for any firm  $i$  and any worker  $j$ . It follows in the market game described below that an “offer” of the (default) bundle 0 by a firm to a worker guarantees that this worker will not be employed by that firm. The preference relation  $P_j$  of each worker  $j$  is taken to be common knowledge among the market participants.

The preferences of the firms, however, are not commonly known. The uncertainty they face at different stages of their market interactions requires us to define their preferences at three different

levels of information. In reverse chronological order we have: the *ex post* level; the *interim* level; and, the *ex ante* level.

Firm  $i$ 's private information is encoded in a (measurable) signal space  $(\Omega_i, \Sigma_i)$ , such that each  $\Omega_i$  is a separable and complete metric space, and  $\Sigma_i$  is the Borel  $\sigma$ -algebra. Let  $\Omega = \times_i \Omega_i$  denote the product of the  $\Omega_i$ 's, and let  $\Sigma$  denote the (product) Borel  $\sigma$ -algebra.

Let  $\mathcal{R}$  denote the (finite) set of complete and transitive binary relations defined on  $\Theta$ . Given the profile of signals  $\omega$  in  $\Omega$ , each firm  $i$ 's *ex post* preferences over outcomes  $\Theta$  is given by  $R_i^\omega \in \mathcal{R}$ . We assume that the realization of the signal  $\omega_i$  fully parametrizes (or reveals) the *ex post* preferences of firm  $i$ , that is,  $R_i^\omega = R_i^{\omega'}$  whenever  $\omega_i = \omega'_i$ . We also assume that these preferences are measurable, that is, for each  $R \in \mathcal{R}$ , the set  $T_i(R) = \{\omega_i \in \Omega_i : R = R_i^\omega, \text{ for some } \omega_{-i} \in \Omega_{-i}\}$  is in  $\Sigma_i$ .

Let  $\pi$  be a measure on  $\Sigma$ . In this setting, this probability measure  $\pi$  should not be interpreted as a (common) prior belief; rather each of its marginals  $\pi_i$  over  $\Sigma_i$  simply identifies the sets of signal realizations for firm  $i$  that are *shy*. We further assume that each marginal  $\pi_i$  is atomless. Notice, however, that the assumptions we impose on this probability measure does not rule out the possibility of some correlation among the realization of preferences of the firms. At the *interim* level, after receiving the signal realization  $\omega_i$ , firm  $i$  forms a conditional preference  $R_i^{\omega_i}$  over the set  $Y_{-i}$  of  $\Sigma_{-i}$ -measurable functions from  $\Omega_{-i}$  into  $\Theta$ . Finally, at the beginning of the game, each firm  $i$  aggregates these conditional preferences in the form of an *ex ante* preference  $R_i$  over the set  $Y$  of  $\Sigma$ -measurable functions from  $\Omega$  into  $\Theta$ . Let  $P_i^{\omega_i}$  (respectively,  $P_i$ ) denote the strict preference relation derived from  $R_i^{\omega_i}$  (respectively,  $R_i$ ). We make two consistency assumptions on the preferences of the firms. The first is a consistency assumption between the *ex post* and the *interim* preferences. It essentially requires that if there are two signals of firm  $i$ ,  $\omega_i$  and  $\omega'_i$ , that reveal the same *ex post* preferences, then the induced interim preferences,  $R_i^{\omega_i}$  and  $R_i^{\omega'_i}$ , cannot differ. The second assumption is a monotonicity requirement between the *interim* and the *ex ante* preferences. Formally, we assume that the preferences of the firms satisfy:

- (1) If  $R_i^\omega = R_i^{\omega'}$ , then  $R_i^{\omega_i} = R_i^{\omega'_i}$ .
- (2) For any  $y, y' \in Y$ , if  $y(\omega_i, \cdot) R_i^{\omega_i} y'(\omega_i, \cdot)$  for  $\pi_i$ -almost every  $\omega_i$ , then  $y R_i y'$ . Moreover, if additionally  $y(\omega_i, \cdot) P_i^{\omega_i} y'(\omega_i, \cdot)$  for every  $\omega_i \in T$  with  $\pi_i(T) > 0$ , then  $y P_i y'$ .

Finally, we assume that the firms' preferences are continuous with respect to sequences of strategy profiles that converge in measure. More specifically, we assume that the following continuity condition holds:

- (3) Fix a firm  $i$ , and let  $y^n$  and  $z^n$  be two sequences in  $Y$  that converge in  $\pi$  measure to  $y$  and  $z$ , respectively. If  $y^n R_i z^n$  for every  $n$ , then  $y R_i z$ .

The matching game is played as follows. After each firm  $i$  observes  $\omega_i$ , both sides of the market meet, and each firm  $i$  sends an offer of a remuneration bundle to each worker  $j \in W$ . Thus the action space of firm  $i$  is  $\mathcal{B}^W$ , where for each  $a_i \in \mathcal{B}^W$ , the bundle  $a_i(j)$  is the one offered by firm  $i$  to worker  $j$ . For each worker  $j$ , given the action profile  $a \in (\mathcal{B}^W)^N$ , define her set of available actions as  $A_j(a) = \{(0, 0)\} \cup \{(1, a_1(j)), \dots, (N, a_N(j))\}$ . Worker  $j$  simply chooses the (unique) most preferred option from that set according to her strict preference relation  $P_j$ , regardless of what other workers are doing. The resulting outcome is the one corresponding to the accepted offers.

Unlike the interaction among workers, the interaction among the firms is strategic, and can be modeled as a Savage game described by  $(\Omega, A, (F_i, \succsim_i)_{i \in N})$  as follows. In the game played by the firms, the common state space of the firms is  $\Omega$  the product space of signal realizations of the firms.

The action space of each firm is  $A = \mathcal{B}^W$ . The set of strategies  $F_i$  available to firm  $i$  is the set of  $\Sigma$ -measurable functions from  $\Omega$  to  $A$ , with the (informational) restriction that  $f_i(\omega_i, \omega_{-i}) = f_i(\omega_i, \omega'_{-i})$  for every  $\omega_{-i}, \omega'_{-i} \in \Omega_{-i}$ .

Let  $\phi: A^N \rightarrow \Theta$  denote the function that maps a given profile of offers from the firms to an outcome, determined by the decisions of the workers. That is,

$$\phi(a) = \theta \in \Theta,$$

such that for each worker  $j$  in  $W$ ,  $\theta(j) \in A_j(a)$  and

$$\theta(j) P_j \theta' \text{ for all } \theta' \in A_j(a) \setminus \{\theta(j)\}.$$

In this Savage game, given  $f, g \in F = \times_{i=1}^N F_i$ , firm  $i$ 's preferences are given by

$$f \succsim_i g \iff \phi \circ f R_i \phi \circ g.$$

To show that the Savage game played by the firms has an equilibrium it is enough to show it satisfies all of the assumptions A1–A7.

Clearly, assumption A1 is satisfied. To see that assumption A2 holds, suppose that  $(f_i, f_{-i}) \sim_i (g_i, f_{-i}) \succsim_i (h_i, f_{-i})$  for all  $h_i \in F_i$ . Consistency condition (2) implies that  $\phi \circ (g_i(\omega_i), f_{-i}) R_i^{\omega_i} \phi \circ (f_i(\omega_i), f_{-i})$  for  $\pi_i$ -almost every  $\omega_i$ . Thus for any event  $E \in \Sigma_i$ ,  $\phi \circ (g_{iE} f_i(\omega_i), f_{-i}) R_i^{\omega_i} \phi \circ (f_i(\omega_i), f_{-i})$  for  $\pi_i$ -almost every  $\omega_i$ . Applying again the consistency condition (2) yields that  $\phi \circ (g_{iE} f_i, f_{-i}) R_i \phi \circ (f_i, f_{-i})$ , thus  $(g_{iE} f_i, f_{-i}) \succsim_i (f_i, f_{-i})$ , as required. By construction of the strategy sets  $F_i$ , assumption A3 also holds. The continuity assumption A4 is implied by the continuity requirement (3). To see that Assumption A5 is satisfied, take for each firm  $i$  the collection of events  $\mathcal{S}_i^m = \{E \in \Sigma_i: \pi_i(E) \leq \frac{1}{m}\}$ .

In order to check A6, suppose that  $(f_i, f_{-i}) \not\approx_j (g_i, f_{-i})$ , and take any  $h_i \in F_i$ . For each  $m = 1, 2, \dots$  take a partition  $\mathcal{E}^m$  of  $\Omega_i$  such that  $\pi_i(E) = \frac{1}{2^m}$  for every  $E \in \mathcal{E}^m$ . We claim that there exists an  $n$  such that  $(f_i, f_{-i}) \not\approx_j (h_{iE} g_i, f_{-i})$  for every  $E \in \mathcal{E}^n$ . Suppose not, that is, suppose that for each  $m$  there exists  $E^m \in \mathcal{E}^m$  such that  $(f_i, f_{-i}) \approx_j (h_{iE^m} g_i, f_{-i})$ . The sequence of strategy profiles  $(h_{iE^m} g_i, f_{-i})$  converges in measure to  $(g_i, f_{-i})$ . By the continuity assumption (3), it follows that  $(f_i, f_{-i}) \approx_j (g_i, f_{-i})$ , a contradiction. Finally, it is easy to see that A7 holds by defining for each firm  $i$  the set  $X_i$  to be the set of strategies that can be written as follows:

$$f_i(\omega_i) = \sum_{R \in \mathcal{R}} a^R \chi_{T_i(R)}(\omega_i),$$

with  $a^R \in A$ . That is,  $X_i$  is the set of strategies of firm  $i$  that are *constant* on each subset  $T_i(R)$  of  $\Omega_i$ . Since  $\mathcal{R}$ , the set of ex post preferences defined on  $\Theta$  is a finite set, it follows that  $X_i$  is *finitely* distinguished subset of the set of strategies of firm  $i$ . Moreover, for each  $f \in F$ , there exists  $g_i \in X_i$  satisfying  $(g_i, f_{-i}) \succsim_i (f_i, f_{-i})$ , thus A7 is satisfied.

## 5. SUFFICIENT CONDITIONS ON RECURSIVE PAYOFFS

Our purpose in this section is to study a subclass of Savage games in which a player's *ex ante* evaluation of strategy profiles can be expressed as a recursive function of her *interim* utility or payoff. The assumptions that we impose on these games essentially allow for almost arbitrary assessments by a player of other players' strategies. The Savage games associated with these games naturally satisfy A1–A6, and include as special cases Bayesian games, games with multiple priors that admit a recursive representation, and more generally games with preferences over interim payoffs satisfying many conditions studied in the literature on decision making under uncertainty.

Consider a game in *interim utility* form specified as follows:

$$((\Omega_i, \Sigma_i), A_i, \mathbf{V}_i, \mathbf{W}_i)_{i=1}^N,$$

where  $(\Omega_i, \Sigma_i)$  is the measurable space of Player  $i$ 's *types* and  $A_i$  is a compact metric space of Player  $i$ 's *actions*. The space of *type profiles*  $\Omega = \times_{i=1}^N \Omega_i$  has the product algebra  $\Sigma = \otimes_{i=1}^N \Sigma_i$ .

Let  $F_i$  be the set of all  $\Sigma_i$ -measurable *strategies*  $f_i: \Omega_i \rightarrow A_i$ . Player  $i$  is associated with an *interim utility function*  $\mathbf{V}_i: A_i \times F_{-i} \times \Omega_i \rightarrow \mathbb{R}$ , where  $\mathbf{V}_i(a_i, f_{-i}|\omega_i)$  is the interim utility for Player  $i$  whose type is  $\omega_i$  if she chooses action  $a_i$  when the other players are choosing their actions according to the strategy profile  $f_{-i}$ . For any strategy profile  $f \in F$  we write  $\mathbf{V}_i(f)$  for the real-valued function  $\omega_i \mapsto \mathbf{V}_i(f_i(\omega_i), f_{-i}|\omega_i)$ , which we assume is always bounded and  $\Sigma_i$ -measurable.

We call a bounded and  $\Sigma_i$ -measurable real-valued function  $\alpha_i: \Omega_i \rightarrow \mathbb{R}$  an *interim payoff* for Player  $i$ . Player  $i$  has *ex ante* preferences over interim payoffs expressed by the utility function  $\mathbf{W}_i$  that associates with each  $\alpha_i$  an *ex ante* utility  $\mathbf{W}_i(\alpha_i)$  in  $\mathbb{R}$ . The *ex ante* utility  $\mathbf{U}_i(f)$  of Player  $i$  for the strategy profile  $f \in F$  is given by means of the recursive form

$$\mathbf{U}_i(f) = \mathbf{W}_i \circ \mathbf{V}_i(f).$$

An *equilibrium* is a Nash equilibrium of the normal form game  $(F_i, \mathbf{U}_i)_{i=1}^N$ .

We make the following decomposition assumption on payoffs, which is essentially the translation of A2 to this setting.

**B1.** For every  $f_{-i} \in F_{-i}$ , and  $f_i, g_i \in F_i$ , if  $\mathbf{U}_i(f_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}) \geq \mathbf{U}_i(h_i, f_{-i})$  for every  $h_i \in F_i$ , then  $\mathbf{U}_i(f_i E g_i, f_{-i}) = \mathbf{U}_i(f_i, f_{-i})$  for every  $E \in \mathcal{F}_i$ .

We also require that *ex ante* utility over strategy profiles be continuous with respect to an atomless measure.

**B2.** There exists a probability distribution  $\mu: \Sigma \rightarrow [0, 1]$  such that

- (1) All the marginal distributions  $\mu_i: \Sigma_i \rightarrow [0, 1]$  of  $\mu$  are atomless.
- (2) If  $f^n$  is a sequence of strategy profiles that converges  $\mu$ -almost everywhere to  $f$ , then  $\mathbf{U}_i(f^n)$  converges to  $\mathbf{U}_i(f)$  for all  $i$ .

Let  $\bar{A}$  be the disjoint union of the sets  $A_i$  endowed with a consistent metric for which it is compact. For each  $f_i \in F_i$  let  $\bar{f}_i$  be the  $\bar{A}$ -valued function on  $\Omega$  given by  $\bar{f}_i(\omega) = f_i(\omega_i)$ . Define  $\bar{F}_i = \{\bar{f}_i: f_i \in F_i\}$ , which yields the Savage game

$$(\Omega, \bar{A}, (\bar{F}_i, \succsim_i)_{i=1}^N)$$

where  $\succsim_i$  is given by  $\bar{f} \succsim_i \bar{g}$  if and only if  $\mathbf{U}_i(f) \geq \mathbf{U}_i(g)$  for any  $\bar{f}, \bar{g} \in \bar{F}_i$ .

**Proposition 5.1.** *If B1 and B2 hold, then the associated Savage game satisfies A1–A6.*

We now explore properties of players' *ex ante* attitudes towards interim payoffs as embodied in  $\mathbf{W}_i$ , which guarantee that B1 is satisfied, in the presence of B2. Of course,  $\mathbf{W}_i$  only depends on a Player  $i$ 's own type, so behaviorally the properties that we discuss are purely decision theoretic embodying the player's attitudes toward non-strategic uncertainty. In this regard, these properties can be compared to the generalizations of expected utility in the literature.

**Example 5.2** (Dynamically consistent intertemporal payoffs). We say that the payoff  $\mathbf{U}_i$  is dynamically consistent if

$$\mathbf{U}_i(f_i, f_{-i}) \geq \mathbf{U}_i(g_i, f_{-i}) \quad \text{for every } g_i \in F_i$$



implies that

$$\mathbf{V}_i(f_i(\omega_i), f_{-i}|\omega_i) \geq \mathbf{V}_i(g_i(\omega_i), f_{-i}|\omega_i) \quad \text{for } \mu_i\text{-almost every } \omega_i \text{ and every } g_i \in F_i.$$

That is, after Player  $i$  receives partial information about the true state of the world, she will not want to revise a choice she made based on her *ex ante* payoff. We show that dynamically consistent payoffs satisfy B1.

For a given  $f_{-i} \in F_{-i}$ , take any  $f_i, g_i \in F_i$  such that

$$\mathbf{U}_i(f_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}) \geq \mathbf{U}_i(h_i, f_{-i}) \quad \text{for every } h_i \in F_i.$$

Dynamic consistency implies that

$$\mathbf{V}_i(f_i(\omega_i), f_{-i}|\omega_i) = \mathbf{V}_i(g_i(\omega_i), f_{-i}|\omega_i) \quad \text{for } \mu_i\text{-almost every } \omega_i.$$

Let  $E = \{\omega_i : \mathbf{V}_i(f_i(\omega_i), f_{-i}|\omega_i) \neq \mathbf{V}_i(g_i(\omega_i), f_{-i}|\omega_i)\}$ , and take any  $E' \in \Sigma_i$ . Then

$$\mathbf{U}_i(f_{iE'}g_i, f_{-i}) = \mathbf{W}_i(\mathbf{V}_i(f_{iE'}g_i, f_{-i})) = \mathbf{W}_i(\mathbf{V}_i(f_{i(E' \cap E)}g_i, f_{-i})) = \mathbf{U}_i(f_{i(E' \cap E)}g_i, f_{-i}).$$

Since  $\mu_i(E) = 0$ , assumption B2(2) implies that

$$\mathbf{U}_i(f_{i(E' \cap E)}g_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}),$$

which implies the required result.

**Example 5.3** (Strictly monotone utility). Some form of monotonicity is present in nearly all generalizations of expected utility. Using the measure from B2, if  $\mathbf{W}_i$  is strictly monotone for the marginal  $\mu_i$ -pointwise ordering of interim payoffs, then B1 holds.

Let  $\mu$  be the measure from B2. For any interim payoffs write  $\alpha_i \geq \beta_i$  if  $\alpha_i(\omega_i) \geq \beta_i(\omega_i)$  for  $\mu_i$ -almost all  $\omega_i$ . Write  $\alpha_i > \beta_i$  if  $\alpha_i \geq \beta_i$  and  $\alpha_i(\omega_i) > \beta_i(\omega_i)$  over a set of positive  $\mu_i$ -measure. Suppose that  $\mathbf{W}_i$  is strictly monotone, that is,  $\mathbf{W}_i(\alpha_i) \geq \mathbf{W}_i(\beta_i)$  holds whenever  $\alpha_i \geq \beta_i$  and  $\mathbf{W}_i(\alpha_i) > \mathbf{W}_i(\beta_i)$  holds whenever  $\alpha_i > \beta_i$ . We show that B1 holds.

Take  $f_i, g_i \in F_i$  such that  $\mathbf{U}_i(f_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}) \geq \mathbf{U}_i(h_i, f_{-i})$  for every  $h_i \in F_i$ , and  $E \in \Sigma_i$ . For interim payoffs  $\alpha_i$  and  $\beta_i$  denote by  $\alpha_i \vee \beta_i$  and  $\alpha_i \wedge \beta_i$  the state-wise supremum and infimum payoffs, and notice that strict monotonicity implies that  $\mathbf{V}_i(f_i, f_{-i}) \wedge \mathbf{V}_i(g_i, f_{-i}) = \mathbf{V}_i(f)$   $\mu_i$ -almost everywhere. (Otherwise, for  $E' = \{\omega_i : \mathbf{V}_i(f_i(\omega_i), f_{-i}|\omega_i) > \mathbf{V}_i(g_i(\omega_i), f_{-i}|\omega_i)\}$ ,  $\mathbf{V}_i(f_{iE'}g_i, f_{-i}) = \mathbf{V}_i(f_i, f_{-i}) \vee \mathbf{V}_i(g_i, f_{-i}) > \mathbf{V}_i(f) \wedge \mathbf{V}_i(g_i, f_{-i})$ , which implies that  $\mathbf{U}_i(f_{iE'}g_i, f_{-i}) > \mathbf{U}_i(f) \wedge \mathbf{U}_i(g_i, f_{-i})$ .) Therefore,

$$\mathbf{U}_i(f) \geq \mathbf{U}_i(f_{iE}g_i, f_{-i}) = \mathbf{W}_i(\mathbf{V}_i(f_{iE}g_i, f_{-i})) \geq \mathbf{W}_i(\mathbf{V}_i(f_i, f_{-i}) \wedge \mathbf{V}_i(g_i, f_{-i})) = \mathbf{W}_i(\mathbf{V}_i(f)) = \mathbf{U}_i(f),$$

which implies the required result.

**Example 5.4** (Supermodular utilities). In the presence of ambiguity aversion, preferences over interim payoffs need not be strictly monotonic though weak monotonicity can usually be guaranteed (see for example Gilboa (1987)). However, if  $\mathbf{W}_i$  can be represented by a Choquet integral and exhibits Schmeidler's (1989) notion of uncertainty aversion, then  $\mathbf{W}_i$  will be supermodular (Denneberg, 1994, Corollary 13.4, p. 161). In fact, weak monotonicity and supermodularity together imply that condition B1 holds.

For any interim payoffs  $\alpha_i$  and  $\beta_i$  denote by  $\alpha_i \vee \beta_i$  and  $\alpha_i \wedge \beta_i$  the state-wise supremum and infimum payoffs. Assume that  $\mathbf{W}_i$  is non-decreasing in the sense that if  $\alpha_i(\omega_i) \geq \beta_i(\omega_i)$  for all  $\omega_i$ , then  $\mathbf{W}_i(\alpha_i) \geq \mathbf{W}_i(\beta_i)$ . Now suppose that  $\mathbf{W}_i$  satisfies supermodularity:

$$\mathbf{W}_i(\alpha_i \vee \beta_i) + \mathbf{W}_i(\alpha_i \wedge \beta_i) \geq \mathbf{W}_i(\alpha_i) + \mathbf{W}_i(\beta_i).$$

We show that B1 holds.

Take  $f_i, g_i \in F_i$  such that  $\mathbf{U}_i(f_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}) \geq \mathbf{U}_i(h_i, f_{-i})$  for every  $h_i \in F_i$ , and  $E \in \Sigma_i$ . Notice that supermodularity implies that  $\mathbf{W}_i(\mathbf{V}_i(f_i, f_{-i}) \wedge \mathbf{V}_i(g_i, f_{-i})) = \mathbf{W}_i(\mathbf{V}_i(f))$ . Therefore,  $\mathbf{U}_i(f) \geq \mathbf{U}_i(f_i E g_i, f_{-i}) = \mathbf{W}_i(\mathbf{V}_i(f_i E g_i, f_{-i})) \geq \mathbf{W}_i(\mathbf{V}_i(f_i, f_{-i}) \wedge \mathbf{V}_i(g_i, f_{-i})) = \mathbf{W}_i(\mathbf{V}_i(f)) = \mathbf{U}_i(f)$ , as required.

**Example 5.5** (Decomposable choice). Moving away from explicit monotonicity, we give a “betweenness” condition on preferences over interim payoffs that generalizes Savage’s P2 postulate and that also satisfies the property B1 above. Consider the *decomposable choice* property of Grant, Kajii, and Polak (2000), which in this setting translates to the following condition:

**GKP:** For any interim payoffs  $\alpha_i, \beta_i$  and events  $E \in \Sigma_i$  if  $\mathbf{W}_i(\alpha_i) > \mathbf{W}_i(\beta_i E \alpha_i)$  and  $\mathbf{W}_i(\alpha_i) \geq \mathbf{W}_i(\alpha_i E \beta_i)$ , then  $\mathbf{W}_i(\alpha_i) > \mathbf{W}_i(\beta_i)$ .

To see that this condition satisfies B1, take  $f_i, g_i \in F_i$  such that  $\mathbf{U}_i(f_i, f_{-i}) = \mathbf{U}_i(g_i, f_{-i}) \geq \mathbf{U}_i(h_i, f_{-i})$  for every  $h_i \in F_i$ . Notice that if  $\mathbf{W}_i(\mathbf{V}_i(f)) = \mathbf{W}_i(\mathbf{V}_i(g_i, f_{-i})) \geq \mathbf{W}_i(\mathbf{V}_i(f_i E g_i, f_{-i}))$  for all  $E \in \Sigma_i$ , then it cannot be the case that  $\mathbf{W}_i(\mathbf{V}_i(f)) > \mathbf{W}_i(\mathbf{V}_i(g_i E f_i, f_{-i}))$  for  $E \in \Sigma_i$ .

**Example 5.6.** [Recursive payoffs with multiple priors] Suppose that Player  $i$  has a bounded measurable *payoff function*

$$u_i: A \times \Omega \rightarrow \mathbb{R},$$

where the set of *action profiles*  $A$  is endowed with the product Borel algebra and  $A \times \Omega$  also has the product algebra. Suppose further that for each  $i$ , we are given a set  $D_i$  of probability measures on  $\Sigma$ . For each  $\pi_i \in D_i$  we write  $\hat{\pi}_i$  for its marginal distribution on  $\Sigma_i$  and for the distribution conditional on own types we write  $\pi_i(\cdot | \cdot): \Sigma_{-i} \times \Omega_i \rightarrow [0, 1]$ , whereby  $\pi_i(\cdot | \omega_i)$  is a probability distribution on  $\Sigma_{-i}$  interpreted as the conditional probability distribution on  $\omega_i \in \Omega_i$  realizing.<sup>10</sup> Player  $i$ ’s *ex ante* utilities are of the multiple prior form of Gilboa and Schmeidler (1989) and that  $D_i$  satisfies the *rectangularity* property of Epstein and Schneider (2003):

$$\begin{aligned} \mathbf{U}_i(f) &= \inf_{\pi_i \in D_i} \int_{\Omega} u_i(f(\omega), \omega) d\pi_i(\omega) \\ &= \inf_{\pi_i \in D_i} \int_{\Omega_i} \inf_{\nu_i \in D_i} \int_{\Omega_{-i}} u_i(f(\omega), \omega) d\nu_i(\omega_{-i} | \omega_i) d\hat{\pi}_i(\omega_i). \end{aligned}$$

With this separation we let

$$\mathbf{V}_i(a_i, f_{-i} | \omega_i) = \inf_{\nu_i \in D_i} \int_{\Omega_{-i}} u_i(a_i, f_{-i}(\omega_{-i}), \omega) d\nu_i(\omega_{-i} | \omega_i),$$

for each  $a_i \in A_i$ ,  $f_{-i} \in F_{-i}$  and  $\omega_i \in \Omega_i$ . For any interim payoff  $\alpha_i$  we let

$$\mathbf{W}_i(\alpha_i) = \inf_{\pi_i \in D_i} \int_{\Omega_i} \alpha_i(\omega_i) d\hat{\pi}_i(\omega_i).$$

We make three assumptions.

**C1.** For each  $\omega \in \Omega$ , the function  $a \mapsto u_i(a, \omega)$  is continuous.

<sup>10</sup>The existence of such a conditional distribution is always guaranteed when the underlying probability space is a Radon space. We note that when each  $D_i$  is a singleton and we are in a Bayesian game setting, the existence of equilibrium result in this section does not require the decomposability of priors into marginals and conditionals.

**C2.** There is a probability measure  $\mu: \Sigma \rightarrow [0, 1]$  such that

- (1) Each  $\pi \in \cup_i D_i$  is absolutely continuous with respect to  $\mu$ .
- (2) The marginal distributions of  $\mu$  over each  $\Sigma_i$  are all atomless.
- (3) For each  $i$  the set  $D_i$  is weak\* compact in the dual of  $L_\infty(\Omega, \Sigma, \mu)$ , the space of real valued  $\mu$ -essential bounded (equivalence classes) functions on  $\Omega$ .

**C3.** The marginal densities  $\{\hat{\pi}_i: \pi_i \in D_i\}$  are mutually absolutely continuous.

Epstein and Marinacci (2007) characterize this last condition for the maxmin expected utility form in terms of a condition of Kreps (1979).

**Proposition 5.7.** *If the game with multiple priors satisfies C1, C2, and C3, then the associated game in interim utility form satisfies B1 and B2.*

## 6. LOCATION GAMES WITH RECURSIVE PAYOFFS

We now use this convenient recursive structure to investigate two examples of location games on the sphere. The first is a Bayesian game with payoffs and individual priors that depend on the full profile of types. In the second game, players have recursive payoffs with multiple priors as in example 5.6.

**Bayesian location game.** Consider the  $N$ -player Bayesian game

$$((\Omega_i, \Sigma_i), A_i, u_i, \nu_i)_{i=1}^N,$$

where  $(\Omega_i, \Sigma_i)$ , the measurable space of Player  $i$ 's types, is the unit interval  $[0, 1]$ . The action space  $A_i$  of each player is the unit sphere  $\mathbb{S}^n$  in  $\mathbb{R}^{n+1}$ . Player  $i$ 's prior  $\nu_i$  is a probability density function  $\nu_i: \Omega \rightarrow \mathbb{R}_+$ , which has full support and is Lipschitz continuous.

Let  $\mathbb{B}^{n+1}$  denote the unit ball of  $\mathbb{R}^{n+1}$ . The payoff function  $u_i: A \times \Omega \rightarrow \mathbb{R}$  of Player  $i$  is given by

$$u_i(a, \omega) = \gamma_i \|P_i(a_i, \omega_i) - R_i(a_{-i}, \omega)\|^2 + (1 - \gamma_i) \|P_i(a_i, \omega_i) - Q_i(\omega_i)\|^2$$

with Lipschitz continuous functions  $P_i: A_i \times \Omega_i \rightarrow \mathbb{S}^n$ ,  $Q_i: \Omega_i \rightarrow \mathbb{S}^n$ ,  $R_i: A_{-i} \times \Omega \rightarrow \mathbb{B}^{n+1}$ , and  $0 \leq \gamma_i < \frac{1}{2}$ . We interpret  $R_i(a_{-i}, \omega)$  as Player  $i$ 's idiosyncratic way of calculating the (generalized) average of the other players' locations,  $Q_i(\omega_i)$  as her most preferred location given her type  $\omega_i$ , and  $P_i(a_i, \omega_i)$  as a (possible, but not required) distortion induced by her type  $\omega_i$  on the degree of her desire to be close to the other players' expected location and her own preferred location. In particular, we allow that Player  $i$  may be "social" for some types, for example,  $P_i(a_i, \omega_i) = -a_i$ , but may be "anti-social" for other types, for example,  $P_i(a_i, \omega_i) = a_i$ . We assume that the inverse correspondence  $P_i^{-1}: A_i \times \Omega_i \rightarrow A_i$  defined by

$$P_i^{-1}(a_i, \omega_i) = \{a'_i \in A_i: a_i = P_i(a'_i, \omega_i)\},$$

is non-empty valued and Lipschitz continuous with constant  $K$ . That is, for all  $x, y \in A_i \times \Omega_i$  we have

$$\delta(P_i^{-1}(x), P_i^{-1}(y)) \leq K\|x - y\|,$$

where  $\delta$  is the Hausdorff distance between sets in  $\mathbb{R}^{n+1}$ .

We shall show that this game has a Bayesian Nash equilibrium (in pure strategies). Clearly, B1 and B2 hold. By Proposition 5.1 we need only show that A7 is satisfied. Fix Player  $i$ , a strategy

profile  $f_{-i}$  of other players, and a type  $\omega_i \in \Omega_i$ . For each action  $a_i$ , let  $\mathbf{V}_i(a_i, f_{-i}|\omega_i)$  be the *interim expected utility*

$$\mathbf{V}_i(a_i, f_{-i}|\omega_i) = \int_{\Omega_{-i}} u_i(a_i, f_{-i}(\omega), \omega) \nu_i(\omega_{-i}|\omega_i) d\lambda(\omega_{-i}),$$

where  $\lambda$  is the Lebesgue probability measure on  $[0, 1]^{N-1}$  and

$$\nu_i(\omega_{-i}|\omega_i) = \frac{\nu_i(\omega)}{\int_{\Omega_{-i}} \nu_i(\omega) d\lambda(\omega_{-i})}$$

is the conditional probability density of  $\nu_i$  on  $\Omega_{-i}$ .

Let

$$M_i(a_{-i}, \omega) = \gamma_i R_i(a_{-i}, \omega) + (1 - \gamma_i) Q_i(\omega_i),$$

and

$$m_i(f_{-i}|\omega_i) = \int_{\Omega_{-i}} M_i(f_{-i}(\omega_{-i}), \omega) \nu_i(\omega_{-i}|\omega_i) d\lambda(\omega_{-i}).$$

We see that

$$\begin{aligned} u_i(a, \omega) &= \|P_i(a_i, \omega_i)\|^2 - 2\langle P_i(a_i, \omega_i), M_i(a_{-i}, \omega) \rangle \\ &\quad + \gamma_i \|R_i(a_{-i}, \omega)\|^2 + (1 - \gamma_i) \|Q_i(\omega_i)\|^2, \end{aligned}$$

where for any  $x, y \in \mathbb{R}^{n+1}$ ,  $\langle x, y \rangle \in \mathbb{R}$  is the inner-product. Thus

$$\begin{aligned} \mathbf{V}_i(a_i, f_{-i}|\omega_i) &= \|P_i(a_i, \omega_i) - m_i(f_{-i}|\omega_i)\|^2 + \|m_i(f_{-i}|\omega_i)\|^2 \\ &\quad + \int_{\Omega_{-i}} \gamma_i \|R_i(f_{-i}(\omega_{-i}), \omega)\|^2 \nu_i(\omega_{-i}|\omega_i) d\lambda(\omega_{-i}) + (1 - \gamma_i) \|Q_i(\omega_i)\|^2. \end{aligned}$$

Noting that  $\|m_i(f_{-i}|\omega_i)\| \geq 1 - 2\gamma_i > 0$ , define the point

$$q_i(f_{-i}|\omega_i) = \frac{-m_i(f_{-i}|\omega_i)}{\|m_i(f_{-i}|\omega_i)\|},$$

which is the point on the sphere that is farthest away from  $m_i(f_{-i}|\omega_i)$ .

Any  $a_i \in A_i$  satisfying  $q_i(f_{-i}|\omega_i) = P_i(a_i, \omega_i)$ , equivalently,  $a_i \in P_i^{-1}(q_i(f_{-i}|\omega_i), \omega_i)$ , maximizes  $\mathbf{V}_i(\cdot, f_{-i}|\omega_{-i})$ . In particular, any strategy  $f_i^*$  satisfying

$$P_i(f_i^*(\omega_i), \omega_i) = q_i(f_{-i}|\omega_i) \quad \text{equivalently} \quad f_i^*(\omega_i) \in P_i^{-1}(q_i(f_{-i}|\omega_i), \omega_i)$$

for all  $\omega_i$  is a best response for Player  $i$  to  $f_{-i}$ .

Now  $\omega \mapsto \nu_i(\omega_{-i}|\omega_i)$  is a Lipschitz continuous function because the prior  $\nu_i$  is a Lipschitz continuous function that is bounded away from zero. Therefore,  $\omega_i \mapsto m_i(f_{-i}|\omega_i)$  is also a Lipschitz continuous function with Lipschitz constant  $K'$  that is independent of the choice of  $f_{-i}$  because of the Lipschitz continuity of  $Q_i$  and  $R_i$ . This in turn implies that  $\omega_i \mapsto q_i(f_{-i}|\omega_i)$  is a Lipschitz continuous function with Lipschitz constant  $K''$ , independent of the choice of  $f_{-i}$ , because  $\gamma_i < \frac{1}{2}$ . Finally, we conclude that the closed non-empty valued correspondence  $\omega_i \mapsto P_i^{-1}(q_i(f_{-i}|\omega_i), \omega_i)$  is Lipschitz continuous with some constant  $K^*$  that is the same for all  $f_{-i}$ . By the theorem of Kupka (2005), this correspondence with one dimensional domain has a  $K^*$ -Lipschitz continuous selection  $\hat{f}_i$ , which is a best response to  $f_{-i}$ . Let  $X_i$  be the family of  $K^*$ -Lipschitz continuous strategies for Player  $i$ . By the Arzelà-Ascoli compactness theorem assumption A7 is satisfied.

**Location game with multiple-priors.** Consider another  $N$ -player location game in which once again Player  $i$ 's type space  $\Omega_i$  is  $[0, 1]$  and her action space  $A_i$  is the unit sphere  $\mathbb{S}^n$ . The player's payoff function  $u_i: A \times \Omega \rightarrow \mathbb{R}$  is

$$u_i(a, \omega) = \begin{cases} \|a_i - M_i(a_{-i}, \omega)\|^2, & \text{if } \min_{j \neq i} \omega_j \leq \frac{1}{2}, \\ 1, & \text{otherwise;} \end{cases}$$

where  $M_i: A_{-i} \times \Omega \rightarrow \mathbb{S}^n$  is a Lipschitz continuous function. If the type of at least one of the players other than  $i$  is less than or equal to one-half, that is *Low*, then Player  $i$  wishes to locate on the circle as far away as possible from  $M_i(a_{-i}, \omega)$ , and may get a payoff greater than one. However, if the type of every player aside from  $i$  is greater than a half, that is *High*, then Player  $i$  has a guaranteed payoff 1.

We assume the preferences of Player  $i$  over strategy profiles take the maxmin expected utility or "multiple priors" form of Gilboa and Schmeidler (1989). For each  $i$ , let  $\lambda_i$  be the Lebesgue distribution on  $\Omega_i$ , and let  $\lambda$  be the product distribution on  $\Omega$ . Let  $\hat{D}_i$  be a weakly compact set of probability density functions on  $\Omega_i$  in which each  $\hat{\nu}_i$  in  $\hat{D}_i$  is mutually absolutely continuous with  $\lambda_i$ . Let  $\mu_{-i}: \Omega_{-i} \times \Omega_i \rightarrow \mathbb{R}_+$  and  $\nu_{-i}: \Omega_{-i} \times \Omega_i \rightarrow \mathbb{R}_+$  be functions for which  $\omega_i \mapsto \mu_{-i}(\cdot | \omega_i)$  and  $\omega_i \mapsto \nu_{-i}(\cdot | \omega_i)$  are mappings to conditional probability densities on  $\Omega_{-i}$ . We assume that for each fixed  $\omega_{-i}$  the function  $\mu_{-i}$  is Lipschitz continuous in  $\omega_i$ . We also assume that if  $\omega_{-i}$  is in the support of  $\mu_{-i}(\cdot | \omega_i)$ , then at least one player is of *Low* type. We also assume that for each  $\omega_i$ , the support of  $\nu_{-i}(\cdot | \omega_i)$  is a subset of  $(\frac{1}{2}, 1]^{N-1} \subseteq \Omega_{-i}$ .

Now take  $D_i$  to be the following set of probability densities defined on  $\Omega$ :

$$D_i = \{\omega \mapsto \pi_i(\omega_i)(\alpha \mu_{-i}(\omega_{-i} | \omega_i) + (1 - \alpha) \nu_{-i}(\omega_{-i} | \omega_i)) : \pi_i \in \bar{D}_i, 0 \leq \alpha \leq 1\}.$$

The *ex ante* utility of Player  $i$  for the strategy profile  $f \in F$  is

$$U_i(f) = \min_{\pi \in D_i} \int_{\Omega} u_i(f(\omega), \omega) \pi(\omega) d\lambda(\omega).$$

We show that this game has an equilibrium. Since  $D_i$  satisfies the *rectangularity* property this is a game in interim form satisfying C1, C2, C3. By Proposition 5.7 we need only show that A7 is satisfied.

Since  $D_i$  satisfies the *rectangularity* property it follows if  $\omega_i$  is realized for Player  $i$  then the player wants to maximize the interim utility which in this case is given by

$$\mathbf{V}_i(a_i, f_{-i} | \omega_i) = \min_{\pi_{-i} \in \{\mu_{-i}(\cdot | \omega_i), \nu_{-i}(\cdot | \omega_i)\}} \int_{\Omega_{-i}} u_i(a_i, f_{-i}(\omega_{-i}), \omega) \pi_{-i}(\omega_{-i}) d\lambda_{-i}(\omega_{-i}).$$

For each  $\omega_i$  and  $f_{-i}$  let  $m_i(f_{-i} | \omega_i)$  be the point

$$m_i(f_{-i} | \omega_i) = \int_{\Omega_{-i}} M_i(f_{-i}(\omega_{-i}), \omega) \mu_{-i}(\omega_{-i} | \omega_i) d\lambda(\omega_{-i}).$$

There is a  $K$  that is independent of  $f_{-i}$  such that  $\omega_i \mapsto m_i(f_{-i} | \omega_i)$  is  $K$ -Lipschitz continuous.

Fixing  $\omega_i$  and  $f_{-i}$  we notice that for any  $a_i$  we have

$$\begin{aligned} & \int_{\Omega_{-i}} u_i(a_i, f_{-i}(\omega_{-i})) \mu_i(\omega_{-i} | \omega_i) d\lambda(\omega_{-i}) \\ &= \int_{\Omega_{-i}} \|a_i - M_i(f_{-i}(\omega_{-i}), \omega)\|^2 \mu_i(\omega_{-i} | \omega_i) d\lambda(\omega_{-i}) \\ &\geq 1 = \int_{\Omega_{-i}} u_i(a_i, f_{-i}(\omega_{-i})) \nu_i(\omega_{-i} | \omega_i) d\lambda(\omega_{-i}) \end{aligned}$$

if and only if

$$\|a_i - m_i(f_{-i} | \omega_i)\|^2 \geq \|m_i(f_{-i} | \omega_i)\|^2.$$

But there is always a point in  $\mathbb{S}^n$  satisfying  $\|a_i - m_i(f_{-i} | \omega_i)\| \geq \|m_i(f_{-i} | \omega_i)\|$ . Therefore, the value  $\mathbf{V}_i(a_i, f_{-i} | \omega_i)$  of such a point  $a_i$  is one. But the maximum of  $\mathbf{V}_i(a_i, f_{-i} | \omega_i)$  is also one. From this we conclude that  $a_i$  maximizes  $\mathbf{V}_i(a_i, f_{-i} | \omega_i)$  if and only if  $\|a_i - m_i(f_{-i} | \omega_i)\| \geq \|m_i(f_{-i} | \omega_i)\|$ . In particular, the maximizers of  $\mathbf{V}_i$  have the following form:

$$B_i(f_{-i} | \omega_i) = \arg \max_{a_i \in A_i} \mathbf{V}_i(a_i, f_{-i} | \omega_i) = \{a_i \in A_i : \langle a_i, m_i(f_{-i} | \omega_i) \rangle \leq \frac{1}{2}\},$$

for each  $\omega_i$  and  $f_{-i}$ .

This is an upper hemicontinuous correspondence from  $[0, 1]$  to  $\mathbb{S}^n$ . That is, there is a  $K^*$  independently of  $f_{-i}$  such that

$$\delta(B_i(f_{-i} | \omega_i), B_i(f_{-i} | \omega'_i)) \leq K^* |\omega_i - \omega'_i|,$$

for all  $\omega_i, \omega'_i$ , where  $\delta$  is the Hausdorff distance between sets. By the theorem of Kupka (2005), this correspondence has a  $K^*$ -Lipschitz continuous selection  $\hat{f}_i$ . Once again applying the Arzelà-Ascoli theorem yields the desired result.

## 7. CONCLUDING REMARKS AND RELATED WORK

We conclude with a discussion of some related models as well as issues arising from the results we have derived in the framework of Savage games.

**Strategic interaction between non-expected utility maximizers.** The literature on modeling strategic interactions between individuals whose preferences may not conform to expected utility theory has mostly focused on normal-form games of complete information. Hence the only uncertainty the players face is with regard to the strategy choices of their opponents. Azrieli and Teper (2011) provide a very nice summary of this literature, comparing and contrasting the different approaches that have been taken.<sup>11</sup>

In their paper, however, Azrieli and Teper consider a class of games with incomplete information where the payoff of a player can depend on the state of nature as well as the profile of actions chosen by the players.<sup>12</sup> Each player can choose any state-contingent *randomization* over her set of available actions that is measurable with respect to her information which in turn is characterized by a partition of a *finite* state space. They allow for players whose ex ante preferences over strategy profiles are generated by (fairly) arbitrary functionals defined on the state-contingent payoffs associated with each strategy profile. Assuming these functionals satisfy standard continuity and

<sup>11</sup>Rather than reproduce their discussion here we refer the interested reader to section 1.2 on page 311 of their paper.

<sup>12</sup>Kajii and Ui (2005) also model a class of games with incomplete information for the particular class of preferences that admit a multiple prior representation.

monotonicity properties, their main result is that an *ex ante* equilibrium exists in every game if and only if these functionals are quasi-concave.

By enriching the strategy space so that players can choose (consistent with their respective information) state-contingent *randomizations* over their actions, allows Azrieli and Teper to establish the existence of an *ex ante* equilibrium in an incomplete information game with a finite state space, just as Anscombe and Aumann (1963) were able to characterize the class of preferences that admit a subjective expected utility representation in a setting with a finite state space. We note, however, that this is achieved at the cost of assuming that players have access to objective randomizing devices, a significant departure from the Savage approach in which all uncertainty is subjective.

Azrieli and Teper also note that the quasi-concavity of the functionals that generate the players' *ex ante* preferences over strategy profiles, can readily be related to the property of ambiguity aversion in the Anscombe and Aumann two-stage setting where the subjective uncertainty is resolved first followed by an objective randomization over the final consequences. However, as Eichberger and Kelsey (1996) argue, there is no natural counterpart interpretation in Savage's setting of purely subjective uncertainty. Hence we do not find it surprising that for any Savage game with which there is associated a game in *interim* utility form, properties B1 and B2 neither imply nor require the quasi-concavity of the *ex ante* utility  $\mathbf{W}_i$  from section 5 above.

**Universal state space.** One question that we do not attempt to answer in this paper is whether it is possible to construct a state space that is a comprehensive representation of the uncertainty faced by players, in the sense of Mertens and Zamir (1985) and Brandenburger and Dekel (1993). We note that Epstein and Wang (1996) do provide such foundations for a setting with purely subjective uncertainty and where the preferences of players need not conform to subjective expected utility theory and so may exhibit non-neutral attitudes toward ambiguity. However, Epstein and Wang's setting does not allow for interdependent preferences. Bergemann, Morris, and Takahashi (2011) construct a universal type space for players with interdependent preferences, but as their framework explicitly involves objective randomization, it is not clear to us how their analysis could be conducted in a Savage setting of purely subjective uncertainty. Finally, Di Tillio (2008) allows for more general preferences, albeit in a setting in which there is only a finite number of outcomes.

**Rationalizability.** It is also not clear to us what is the appropriate notion of rationalizability in the framework of Savage games. There is an extensive literature that provides foundations for equilibrium in terms of rationalizable behavior, see for example Brandenburger and Dekel (1987) in the context of subjective uncertainty, Tan and Werlang (1988) and Börgers (1993). However, in many of these papers, rationality is expressed in terms of "state-independent expected utility."<sup>13</sup>

To allow for state-dependent ordinal preferences, an alternative notion of rationalizability is needed. As noted by Morris and Takahashi (2012), rationalizability defined in terms of ordinal preferences is invariant to the choice of state space, unlike rationalizability defined in terms of expected utility. However, Morris and Takahashi's notion of rationalizability requires explicit randomization of the kind implied by Anscombe–Aumann acts, which is not available in our setting. Epstein (1997) investigates rationalizability in a setting where strategies are analogs of Savage style acts, nevertheless he rules out state-dependent preferences and restricts the analysis to finite normal form games.

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<sup>13</sup>An exception is Tan and Werlang (1988), who start with a Bayesian game in which players have a state-dependent subjective expected utility function.

**Purification of mixed strategies.** We have entirely avoided any assumption on the independence or near independence of player information or types or payoffs. Indeed, in our Bayesian game example, types are statistically dependent via arbitrary Lipschitz-continuous probability density functions. This is in stark contrast with the purification results that follow the classical work of Dvoretzky, Wald, and Wolfowitz (1950), Radner and Rosenthal (1982) and Milgrom and Weber (1985), and related literature.<sup>14</sup> One interpretation of this difference is that while decomposability arguments are also at the heart of purification techniques, those require purification of objectively randomized equilibria. The present paper highlights how our use of the decomposition property can be interpreted as purification of a purely subjective kind.

An important open question is whether it is possible to obtain our results even for standard Bayesian games with interdependent priors using the purification techniques of the extant literature. That literature has focused on the existence of pure-strategy equilibrium in Bayesian games in which information is diffuse. The usual approach is to identify conditions on the information structure of the game that allows us to find a profile of pure strategies that is payoff equivalent to any given equilibrium (randomized) strategy profile. To the best of our knowledge, the techniques that have been developed so far rule out interdependent payoffs and require independent distributions of types.

#### APPENDIX A. PROOFS

*Proof of Proposition 3.1.* Suppose that  $f \sim_i (g_i, f_{-i}) \succsim_i (h_i, f_{-i})$  for every  $h_i \in F_i$ . In particular,  $f \sim_i (g_i, f_{-i}) \succsim_i (g_i E f_i, f_{-i})$  for every  $E \in \mathcal{F}_i$ . For any  $E \in \mathcal{F}_i$  we have  $f \succsim_i (g_i \Omega \setminus E f_i, f_{-i})$ , thus, by **P2**,  $f_{i\Omega \setminus E} g_i, f_{-i} \succsim_i (g_i, f_{-i}) \sim_i f$ .  $\square$

*Proof of Proposition 3.2.* It is immediate that  $\mathcal{F}_i$  contains  $\emptyset$  and  $\Omega$ . The other two conditions are obtained by noting that

$$g_{iE \setminus E'} f_i = f_{iE'}(g_{iE} f_i),$$

and

$$g_{iE \cup E'} f_i = g_{iE}(g_{iE'} f_i).$$

With this, A3 guarantees that the countable union of events is an event.  $\square$

*Proof of Corollary 3.3.* Let  $\sigma(F_i)$  be the smallest  $\sigma$ -algebra of subsets of  $\Omega$  for which each strategy  $f_i \in F_i$  is measurable. Clearly,  $\sigma(F_i) \subseteq \Sigma_i \subseteq \mathcal{F}_i$ . Pick  $E \in \mathcal{F}_i$ . Because  $|A| \geq 2$ , there are  $f_i, g_i \in F_i$  such that  $E = \{\omega : g_i E f_i(\omega) \neq f_i(\omega)\}$ , which is in  $\sigma(F_i)$ . Thus,  $\Sigma_i = \mathcal{F}_i$  and A3 holds.  $\square$

*Proof of Proposition 3.6.* Clearly, the empty set is in  $\mathcal{N}_i$ . Let  $E \in \mathcal{N}_i$  and  $E' \in \mathcal{F}_i$  such that  $E' \subseteq E$ . If  $E' \notin \mathcal{N}_i$ , then there are  $f \in F$ ,  $g_i \in F_i$  and  $j \in N$  satisfying

$$(g_{iE'} f_i, f_{-i}) \not\approx_j f.$$

But then

$$((g_{iE'} f_i)_{E'} f_i, f_{-i}) = (g_{iE'} f_i, f_{-i}) \not\approx_j f,$$

which is a contradiction, because  $E$  is shy for Player  $i$ .

Furthermore, if  $E, E' \in \mathcal{N}_i$ , then for any  $g_i \in F_i$ , by transitivity of  $\approx_j$ , we have

$$(g_{iE \cup E'} f_i, f_{-i}) = (g_{iE'}(g_{iE} f_i), f_{-i}) \approx_j (g_{iE} f_i, f_{-i}) \approx_j f,$$

<sup>14</sup>Other results include Balder (1988), Khan and Sun (1995), Khan, Rath, and Sun (2006), Loeb and Sun (2006), Fu, Sun, Yannelis, and Zhang (2007), Fu (2008), Podczeck (2009), Loeb and Sun (2009), Khan and Rath (2009), Wang and Zhang (2012), and Greinecker and Podczeck (2013). Using the same techniques, it is possible to show existence of pure equilibrium when there is a continuum of agents—see Schmeidler (1973), and Mas-Colell (1984).



which tells us that  $E \cup E' \in \mathcal{N}_i$ . Finally, by A3 and A4, for any increasing sequence of shy events  $E^n$ , the union  $E$  is an event, and it must be shy for Player  $i$ .  $\square$

**Proof of Proposition 3.7.** We can assume without loss of generality that each  $\mathcal{S}_i^m$  also has the property that if  $E', E \in \mathcal{F}$  and  $E' \subset E \in \mathcal{S}_i^m$ , then  $E' \in \mathcal{S}_i^m$ .

Fix  $\mathcal{S}_i^m$ . Denote by  $E \ominus E'$  the symmetric difference of any two sets  $E, E' \subseteq \Omega$ . Let

$$\mathcal{R}_i^m = \{E \in \mathcal{R}_i : E \ominus E' \notin \mathcal{N}_i \text{ for all } E' \in \mathcal{S}_i^m\}.$$

**Proposition A.1.** *The following hold true:*

- (1) *If  $E \in \mathcal{R}_i^m$ , then  $E \notin \mathcal{S}_i^m$ .*
- (2)  *$\cup_m \mathcal{R}_i^m = \mathcal{R}_i$ .*
- (3) *If  $E^n$  is an increasing sequence of events whose union  $E$  is in  $\mathcal{R}_i^m$ , then eventually  $E^n$  is in  $\mathcal{R}_i^m$ .*

*Proof.* (1) is obvious because the empty set is in  $\mathcal{N}_i$ .

Turning to (2). Suppose that  $E \in \mathcal{R}_i$ . Suppose by way of contradiction that  $E \notin \mathcal{R}_i^m$  for all  $m$ .  $E$  is associated with  $E^m \in \mathcal{S}_i^m$  such that  $D^m = E \ominus E^m \in \mathcal{N}_i$ . Let  $D = \cup_m D^m$ , which is in  $\mathcal{N}_i$  and we see that  $E \setminus D \subseteq E^m$  for all  $m$ . Thus,  $E \setminus D \in \mathcal{S}_i^m$  for all  $m$ . This implies that  $E \setminus D$  is in  $\mathcal{N}_i$ . Thus,  $E \in \mathcal{N}_i$ , which is a contradiction.

For (3) because  $\mathcal{N}_i$  is a  $\sigma$ -ideal, eventually  $E^n$  is in  $\mathcal{R}_i$ . Now if  $E^n \notin \mathcal{R}_i^m$  then there exists a null event  $D$  such that  $E^n \cap D$  is in  $\mathcal{S}_i^m$  for all  $m$ . By the closedness of  $\mathcal{S}_i^m$   $E \cap D \in \mathcal{S}_i^m$  which is impossible.  $\square$

If all  $\mathcal{R}_i^m$  are empty, then all events are null and the proposition is true trivially. So we can assume that  $\mathcal{R}_i^m$  is not empty for all  $m$ .

**Proposition A.2.** *For each  $m$  there is  $c > 0$  such that*

$$\inf_{E \in \mathcal{N}_i} \max_{\omega \in \Omega} \left( \chi_{\Omega \setminus E} \sum_{k=1}^n \alpha^k \chi_{E^k}(\omega) \right) \geq c$$

for any  $\alpha^1, \dots, \alpha^K \geq 0$  and  $\sum_{k=1}^K \alpha^k = 1$ .

*Proof.* There exists a constant  $c > 0$  such that for any finite sequence  $E^1, E^2, \dots, E^n$  in  $\mathcal{R}_i^m$  we have

$$\max_{\omega \in \Omega} \frac{1}{n} |\{1 \leq k \leq n : \omega \in E^k\}| > c.$$

This implies that

$$\max_{\omega \in \Omega} \frac{1}{n} \sum_{k=1}^n \chi_{E^k}(\omega) > c$$

where  $\chi_E$  is the characteristic function of  $E$ . This in turn implies that

$$\max_{\omega \in \Omega} \sum_{k=1}^n \frac{r^k}{\sum_{k=1}^n r^k} \chi_{E^k}(\omega) = \max_{\omega \in \Omega} \frac{1}{\sum_{k=1}^n r^k} \sum_{k=1}^n r^k \chi_{E^k}(\omega) > c,$$

for every  $r^k \in \mathbb{N}$ ,  $k = 1, \dots, n$ , satisfying  $r^k > 0$  for some  $k$ . We conclude that

$$\max_{\omega \in \Omega} \sum_{k=1}^n \alpha^k \chi_{E^k}(\omega) \geq c,$$

for any convex combination  $\alpha^1, \alpha^2, \dots, \alpha^n \geq 0$ ,  $\sum_{k=1}^n \alpha^k = 1$ . Therefore,

$$(1) \quad \inf_{E \in \mathcal{N}_i} \max_{\omega \in \Omega} \left( \chi_{\Omega \setminus E} \sum_{k=1}^n \alpha^k \chi_{E^k}(\omega) \right) = \inf_{E \in \mathcal{N}_i} \max_{\omega \in \Omega} \sum_{k=1}^n \alpha^k \chi_{E^k \setminus E}(\omega) \geq c,$$

for any convex combination.  $\square$

Let  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  be the ordered vector space of all  $\mathcal{N}_i$ -equivalence classes of  $\mathcal{F}_i$ -measurable bounded functions from  $\Omega$  to  $\mathbb{R}$ . That is,  $f_i: \Omega \rightarrow \mathbb{R}$  is in  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  if it is  $\mathcal{F}_i$ -measurable and bounded, and  $g_i: \Omega \rightarrow \mathbb{R}$  is in the equivalence class  $[f_i]$  if  $\{\omega: f_i(\omega) \neq g_i(\omega)\}$  is in  $\mathcal{N}_i$ .

For each  $f_i \in L_\infty(\mathcal{F}_i | \mathcal{N}_i)$ , let

$$\|f_i\|_\infty = \inf_{E \in \mathcal{N}_i} \sup_{\omega \in \Omega} |\chi_{\Omega \setminus E} f_i(\omega)|,$$

By Proposition 3.6  $\mathcal{N}_i$  is a  $\sigma$ -ideal of  $\mathcal{F}_i$ . Thus  $\|\cdot\|_\infty$  is a norm on  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$ , and with this norm the space is a Banach space. Furthermore,  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  has a canonical ordering whereby  $f_i \geq g_i$  if  $\{\omega: g_i(\omega) > f_i(\omega)\}$  is null. With this vector ordering the Banach space  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  is a Banach lattice with a positive cone  $L_\infty^+(\mathcal{F}_i | \mathcal{N}_i)$  that contains any constant function  $\mathbf{c} = c\chi_\Omega$ ,  $c > 0$ , in its interior.

We list the following result for convenience.

**Proposition A.3.** *There exists  $c > 0$  such that for any  $f$  convex hull  $C^m$  in  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  of  $\{\chi_E: E \in \mathcal{R}_i^m\}$  we have  $\|f\|_\infty \geq c$ . In particular,  $C^m$  is disjoint from  $\frac{1}{2}\mathbf{c} - L_\infty^+(\mathcal{F}_i | \mathcal{N}_i)$ .*

A separating hyperplane argument now tells us that there is a continuous linear functional  $\pi_i^m$  on  $L_\infty(\mathcal{F}_i | \mathcal{N}_i)$  separating the two sets. Because zero is an interior point of one set we see that it is non-negative on  $L_\infty^+(\mathcal{F}_i | \mathcal{N}_i)$  and that for some  $d^m > 0$  we have  $\pi_i^m(f_i) > d^m$  for all  $f_i \in C^m$ .

We can therefore consider  $\pi_i^m$  as a finitely additive measure on  $\mathcal{F}_i$ . It gives a value of zero to each  $E \in \mathcal{N}_i$  and greater than  $d^m$  for each  $E \in \mathcal{R}_i^m$ . By the Hewitt-Yosida decomposition there is a countably additive measure  $\pi_i^{mc}$  and a purely finitely additive measure  $\pi_i^{mf}$  such that

$$\pi_i^m = \pi_i^{mc} + \pi_i^{mf}.$$

Pick  $E \in \mathcal{R}_i^m$ . Because  $\pi_i^{mf}$  is purely finitely additive, for  $d^m > \alpha > 0$  there is an increasing sequence  $E^n \in \mathcal{F}_i$ ,  $\cup_n E^n = E$ , and

$$\lim_n \pi_i^{mf}(E^n) \leq \alpha.$$

But  $E^n \in \mathcal{R}_i^m$  eventually for  $n$  large enough. From this we conclude that for such  $n$

$$\pi_i^{mc}(E) \geq \pi_i^{mc}(E^n) = \pi_i^m(E^n) - \pi_i^{mf}(E^n) \geq \gamma^m - \alpha > 0.$$

Normalize each  $\pi_i^{mc}$  making it a probability measure and consider the probability measure:

$$\pi_i = \frac{1}{2^m} \sum_{m=1}^{\infty} \pi_i^{mc}(E).$$

We see that  $\pi_i(E) > 0$  for each  $E \in \mathcal{R} = \cup_m \mathcal{R}_i^m$ . That is,  $\pi_i$  is the required measure.  $\square$

*Proof of Proposition 3.8.* Suppose that  $E \in \mathcal{R}_i$ . For some player  $j \in N$ , some strategy profile  $f \in F$ , and some strategy  $g_i \in F_i$  we have  $(g_i f_i, f_{-i}) \not\approx_j f$ . Thus, there exists a sequence  $\{E^1, \dots, E^k\} \subseteq \mathcal{F}_i$ ,  $\cup_k E^k = \Omega$ , satisfying  $(f_i E^k(g_i E f_i), f_{-i}) \not\approx_j f$  for all  $k$ . But  $f_i E^k(g_i E f_i) = g_i E \setminus E^k f_i$ . Thus, for all  $k$  the event  $E \setminus E^k$  is in  $\mathcal{R}_i$ . Also, because  $\mathcal{N}_i$  is an ideal there must be some

$k^*$  such that  $E^{k^*} \cap E$  is relevant for Player  $i$ . The relevant events  $E \setminus E^{k^*}$  and  $E^{k^*} \cap E$  are disjoint and their union is  $E$ .  $\square$

**Proof of Proposition 3.9.** We need only show that (2) implies (1). Let  $\pi_i$  be from condition (2). If  $\pi_i(\Omega) = 0$ , then we are done, since there are no relevant events. Otherwise, without loss of generality we can assume  $\pi_i(\Omega) = 1$ .

For each  $m$ , define the set

$$\mathcal{S}_i^m = \{E \in \mathcal{F}_i : \pi_i(E) \leq 1/m\}.$$

Notice that each  $\mathcal{S}_i^m$  is closed and note that  $\bigcap_m \mathcal{S}_i^m \subseteq \mathcal{N}_i$ .

Now let  $E^1, E^2, \dots, E^n$  is a finite sequence in  $\mathcal{F}_i$  not in  $\mathcal{S}_i^m$ . We have

$$\begin{aligned} \max_{\omega \in \Omega} \frac{1}{n} |\{1 \leq k : \omega \in E^k\}| &= \frac{1}{n} \max_{\omega \in \Omega} \sum_{k=1}^n \chi_{E^k} \\ &\geq \frac{1}{n} \int_{\Omega} \sum_{k=1}^n \chi_{E^k}(\omega) \, d\pi_i(\omega) \\ &= \frac{1}{n} \sum_{k=1}^n \pi_i(E^k) \\ &\geq \frac{1}{n} \left(n \frac{1}{m}\right) = \frac{1}{m}. \end{aligned}$$

This shows that A5 holds.

We show that A6 also holds. Suppose that  $(g_i, f_{-i}) \not\approx_j f$ . Suppose by way of contradiction that for each  $m$  there is  $E^m$  satisfying  $\pi_i(E^m) \leq \frac{1}{m}$  such that  $(h_{iE^m} g_i, f_{-i}) \approx_j f$ . Noting that  $\chi_{E^m}$  converges to zero in  $\pi_i$ -measure we can move to a subsequence such that  $\chi_{E^m}$  converges  $\pi_i$ -almost surely to zero. But zeros of  $\pi_i$  are all null for Player  $i$ . Thus, by A4  $(g_i, f_{-i}) \approx_j f$ , which is a contradiction.  $\square$

**A fixed point theorem.** We begin with a statement of a fixed point theorem and apply it to prove Theorem 3.11. For a complete proof of the fixed point theorem used in this subsection, please refer to Meneghel and Tourky (2013).

Let  $(S, \Sigma, \mu)$  be an atomless probability space and let  $T$  be a topological space. Let  $L(S, T)$  be the set of all functions, not necessarily measurable, from  $S$  to  $T$ . Endow  $L(S, T)$  with the topology of pointwise convergence.

A set-valued (possibly empty valued) mapping  $B: F \rightarrow F$  is a *decomposable mapping* if its domain  $F$  and values  $B(f)$ , for all  $f \in F$ , are decomposable sets. A decomposable mapping  $B$  is  $\mu$ -*sequentially closed graphed* if the following hold:

- (1) If  $\mu(E) = 0$  and  $g \in B(f)$ , then  $h_E g \in B(f)$  and  $g \in B(h_E f)$  for all  $h \in F$ .
- (2)  $F$  is sequentially closed in  $L(S, T)$ .
- (3)  $B$  has a sequentially closed graph in  $F \times F$ .

A *fixed point* of  $B$  is a function  $f \in F$  satisfying  $f \in B(f)$ .

**Theorem A.4** (Corollary 2.3, Meneghel and Tourky (2013)). *Let  $B: F \rightarrow F$  be a decomposable  $\mu$ -sequentially closed-graphed mapping. If for a compact and metrizable  $X \subseteq F$  we have  $X \cap B(f) \neq \emptyset$  for each  $f \in F$ , then  $B$  has a fixed point.*

**Proof of Theorem 3.11.** Assume that A7 holds.

For each  $i$  let  $S_i = \Omega$  and  $T_i = A$ . Each  $f \in F$  can be considered a function from  $S$  to  $T$  whereby

$$f(s_1, \dots, s_N) = (f_1(s_1), \dots, f_N(s_N)).$$

For each  $i$  consider the atomless measure space  $(S_i, \mathcal{F}_i, \pi_i)$  from Proposition 3.9. We will assume that at least one player has a relevant event and that all non-zero  $\pi_i$  are probability measures. Let  $\mathcal{F} = \otimes_{i=1}^N \mathcal{F}_i$  be the tensor product. Each  $E$  in  $\mathcal{F}$  that is not the empty set is of the form

$$(E_1, E_2, \dots, E_N),$$

where  $E_i \in \mathcal{F}_i$  for each  $i$ . Now if  $f, g \in F$ , then

$$g_E f(s_1, \dots, s_N) = (g_{1E_1} f_1(s_1), \dots, g_{NE_N} f_N(s_N)),$$

which is in  $F$ .

Let  $\mu: \mathcal{F} \rightarrow [0, 1]$  be the probability measure given by

$$\mu(E) = \frac{1}{N} \sum_{i=1}^N \pi_i(E_i).$$

This is an atomless measure and if  $\mu(E) = 0$ , then each  $E_i$  is null for Player  $i$ . For each  $f \in F$ , let

$$B(f) = \{g \in F: g_i \text{ is a best response to } f_{-i} \text{ for all } i\}.$$

Notice that if  $\mu(E) = 0$  and  $g \in B(f)$  we have  $h_E g \in B(f)$  and  $g \in B(h_E f)$ .

Now our sets  $X_i$  are compact and metrizable in the topology of pointwise convergence. Therefore, their product  $X \subseteq F$  is compact and metrizable in the same topology. Assume first that there is only one player. Clearly, an equilibrium exists because the player maximizes her preferences in the compact and metrizable set  $X$ . Now suppose that there are two or more players. By assumption A7, the sequentially closed, decomposable set  $\tilde{X}_i$  is a subset of  $F_i$  for each  $i$ . Let  $\tilde{X}$  be the product of  $X_i$ , which is sequentially closed and decomposable once again. Let  $\tilde{B}: \tilde{X} \rightarrow \tilde{X}$  be the restriction of  $B$  to  $\tilde{X}$ . We see that  $\tilde{B}$  is a decomposable mapping that is also  $\mu$ -sequentially closed graphed. Applying Theorem A.4 gives us the required equilibrium.  $\square$

*Proof of Proposition 5.1.* Suppose that  $f \in F$  and  $g_i \in F_i$  satisfy

$$\mathbf{U}_i(f) = \mathbf{U}_i(g) \geq \mathbf{U}_i(h_i, f_{-i})$$

for all  $h_i \in F_i$ . By B1 this means that

$$\mathbf{U}_i(g_{iE} f_i, f_{-i}) = \mathbf{U}_i(f)$$

for every  $E \in \Sigma_i$ . Thus, A2 holds. That A3 is satisfied is a consequence of Corollary 3.3. Now  $\mathbf{U}_i$  is continuous for pointwise convergent sequences of strategy profiles by B2. So A4 holds.

If  $A$  has less than two points, then all events are null and A5 and A6 hold trivially. If they have two or more points, then by Corollary 3.3  $\bar{\Sigma}_i = \mathcal{F}_i$ . Now the restriction of  $\mu_i$  to  $\bar{\Sigma}_i$  is atomless and if  $\mu(E) = 0$ , then  $E$  is strategically null for Player  $i$  by (2) of B2. By Proposition 3.9 assumptions A5 and A6 hold.  $\square$

*Proof of Proposition 5.7.* Consider the associated game in interim form. Clearly, B2 holds. For B1 let  $\alpha_i, \beta_i$  be the two interim payoffs. Choose  $\mu_i^* \in D_i$  in

$$\arg \min_{\mu_i \in D_i} \int_{\Omega_i} \alpha_i \vee \beta_i \, d\hat{\mu}_i(\omega_i).$$

We see that

$$\mathbf{U}_i(\alpha_i) \geq \mathbf{U}_i(\alpha_i \vee \beta_i) \geq \int_{\Omega_i} \alpha_i(\omega_i) d\hat{\mu}_i^*(\omega_i) \geq \mathbf{U}_i(\alpha_i).$$

Thus, it must be the case that  $\alpha_i$  and  $\alpha_i \vee \beta_i$  agree  $\mu_i^*$ -almost surely. Similarly,  $\beta_i$  and  $\alpha_i \vee \beta_i$  agree  $\mu_i^*$ -almost surely. This implies that  $\alpha_i$  and  $\beta_i$  agree almost surely for all  $\mu_i \in D_i$ . This implies that  $\mathbf{U}_i(\beta_i E \alpha_i) = \mathbf{U}_i(\alpha_i)$  for all  $E_i \in \Sigma_i$ , as required.  $\square$

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