Randomization and Dynamic Consistency*

Jürgen Eichberger
Alfred Weber Institut,
Heidelberg University.

Simon Grant
School of Economics,
University of Queensland.

David Kelsey
Department of Economics,
University of Exeter.

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Abstract

Raiffa (1961) has suggested that ambiguity aversion will cause a strict preference for randomization. We show, however, that dynamic consistency implies that individuals will be indifferent to *ex ante* randomizations. On the other hand, it is possible for a dynamically-consistent ambiguity averse preference relation to exhibit a strict preference for some *ex post* randomizations. We argue that our analysis throws some light on the Reflection paradox and the paradoxes for the smooth model of ambiguity. We show that these rest on whether the randomizations implicit in the set-up are viewed as being resolved before or after the (ambiguous) uncertainty.

Address for Correspondence David Kelsey, Department of Economics, University of Exeter, Rennes Drive, Exeter EX4 4PU, UK.

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1 Introduction

An ambiguous uncertainty is one for which objective probabilities are not available and individuals cannot or do not assign (conventional) subjective probabilities. Ellsberg (1961) argued that individuals would behave differently when faced with ambiguous uncertainties. However, Raiffa (1961) claimed that individuals can eliminate the effect of ambiguity by randomizing. We shall illustrate his argument in the context of a slight variant of the 3-colour Ellsberg problem.

Consider an urn containing one hundred balls, thirty-three of which are red while the remaining sixty-seven balls are black and yellow in unknown proportions. (Note that unlike the standard 3-colour Ellsberg urn slightly less than one third of the balls are red.) One ball is to be drawn at random from the urn. Let R, (respectively, B or Y) denote a 'bet' on red (respectively, black or yellow) which pays £100 if a red (respectively, black or yellow) ball is drawn from the urn and £0 otherwise. Furthermore, let $\bar{R}$, (respectively, $\bar{B}$ or $\bar{Y}$) denote a 'bet' against red (respectively, black or yellow) which pays £0 if a red (respectively, black or yellow) ball is drawn from the urn and £100 otherwise.

Reasoning analogous to that of Ellsberg (1961) suggests that many individuals might prefer bets where probabilities are more precisely defined and hence display the preferences $R \succ B \sim Y$ and $\bar{R} \succ \bar{B} \sim \bar{Y}$. It is easy to see that there is no subjective probability and no utility which can make these preferences compatible with the subjective expected utility theory (SEU) of Savage (1954).

Raiffa (1961) argues that an individual can avoid such ambiguity by randomizing. For instance, in the problem above, making her choice on whether to bet on either a black or a yellow ball being drawn from the urn, conditional on the flip of a 'fair' coin is supposed to remove any ambiguity the individual may perceive. The (overall) probability of winning is 67/200, independent of the actual proportion of black and yellow balls in the urn. Hence, she is facing only conventional risk. This would seem to suggest that ambiguity aversion causes individuals to strictly prefer such a randomization to either of the two pure acts, B or Y, and indeed, since $67/200 > 33/100$, to the pure act R as well. That is, the individual would display the preferences $B^{\{h\}}Y \succ R$ ($\succ B \sim Y$), where $B^{\{h\}}Y$ denotes the 50:50 randomization between B and Y corresponding to the conditional bet: bet on black if the flipped coin comes up heads, otherwise bet on yellow. As a consequence, if she is allowed to make her bet on the colour of ball drawn the urn conditional on the flip of a 'fair' coin, or, more generally, on the outcome of any 50:50 randomization, then she should not select the act R, the unambiguous pure act.

Eichberger and Kelsey (1996) showed that this logic applies to the convex capacity model of ambiguity by Schmeidler (1989) only in the context of the Anscombe-Aumann (henceforth AA) framework, where outcomes are lotteries and randomizations take place after realizations of states. In this setting, the randomization $B^{\{h\}}Y$ corresponds to a two-stage bet in which if the ball drawn from the urn is black (respectively, yellow) the bet pays £100 if the (subsequent) flip of a fair coin is heads (respectively, tails), otherwise the bet pays nothing. This preference for randomization vanishes, however, when outcomes, acts and the randomization device are modelled in the Savage framework in which the resolution of uncertainty takes place in a single stage.[1] To the contrary, for this model of ambiguity-aversion, decision makers are indifferent to randomization.

The present paper contains a general argument which shows that dynamic consistency implies indifference to randomization. The intuition is as follows. Assume the decision maker is initially indifferent between a pair of ambiguous acts a and b. Let d be an act formed by taking an ex ante $\alpha : (1-\alpha)$ randomization of a and b and suppose that she prefers d to both a and b. Let c be a constant act which is strictly preferred to a and b but is strictly inferior to d. Then before the outcome of the randomizing device is known

[1] See also the discussion in Ghirardato (1997).

the individual prefers d to c. However, after she learns the outcome of the randomizing device she will be holding either a or b, both of which she views as inferior to c. This is a clear violation of dynamic consistency.

In many respects our argument is quite general. We use a Savage framework. Apart from this, our results are model free in the sense that they do not assume any particular functional form for preferences. Our analysis does not assume that the decision maker is ambiguity averse or even perceives ambiguity at all. Thus it also applies to some non-expected utility theories that are probabilistically sophisticated.[2] The results do not depend crucially on the assumption that there is a known probability distribution over the outcomes of the randomizing device. Thus it could apply to an ambiguous randomizing device, such as flipping a coin with an unknown (that is, 'ambiguous') bias.

We believe this is especially intuitive when the result of the randomization is made known to the individual *before* the resolution of the ambiguous bet. Assume the decision maker dislikes the acts a and b due to ambiguity. We argue that it is implausible that flipping a coin before the uncertainty is resolved will make these acts more attractive. After the flip, the decision maker remains exposed to either the ambiguous bet a or the ambiguous bet b as determined by the coin.

Eichberger and Kelsey (1996) also show that in the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989), individuals may (or may not) have a strict preference for randomization. Klibanoff (2001) presents a preference-based notion of a stochastically independent randomizing device. He shows that this notion is not compatible with representing beliefs by a convex capacity. In his view, this provides a motivation for using MEU rather than CEU. If one accepts this argument, a preference for randomization remains a possibility.

[2]Strictly speaking it also applies to subjective expected utility, however in this case, indifference to randomization is obvious.

These differences in preferences about ambiguous acts in the AA and Savage frameworks suggest that the sequence of the realizations of the outcomes of the randomizing device and the states matters. The argument put forward by Raiffa (1961) clearly assumes that the realization of the randomizing device comes first, because the decision maker can condition the choice of the ambiguous act on this outcome. In the AA framework, this randomization over ambiguous acts is treated as a mixture of lotteries which takes place state-wise and, hence, after the realization of states. This point has also been made in a context without ambiguity by Kreps (1988) and, more recently, with ambiguity by Seo (2009) and Wakker (2010).[3]

Most of the literature on ambiguity uses the AA approach as its basic framework. As our discussion of the ex-ante and ex-post view shows, choosing the AA framework and the ex-post notion of randomization over acts is not without loss of generality if one deviates from the expected utility axioms. Anscombe and Aumann (1963) were well aware of this distinction and, in Assumption 2 (p. 201), assume explicitly the equivalence between ex-ante and ex-post randomizations.

Organization of the paper In the next section we state our definitions and present the main results. We then show in section 3 how our framework can help shed light on the debate between Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012) as well as the 'reflection example' of Machina (2009). Our conclusions can be found in Section 4.

[3] See in particular Wakker (2010) sections 4.9 and 10.7.

2 Attitudes Towards Randomization

There is a decision maker who faces uncertainty described by a finite state space S. A generic state is denoted by s. We assume that no states are null.[4] She also has access to a randomizing device, for example, a fair coin, an unbiased roulette wheel or a random number generator. We study whether she would wish to make use of it.

The randomizing device is independent of the uncertainty, about which the decision maker is concerned. Its output, r, is contained in a finite set R. We model the fact that R describes the outcomes of a randomizing device by assuming that there is a probability distribution π defined on R. For simplicity, we assume that for all non-empty $E \subsetneqq R$, $1 > \pi(E) > 0$. Thus, no non-empty subsets of R are null. We believe a decision-maker is unlikely to use a randomizing device for which some of the outcomes are null. To model randomization, we consider the grand state space, which is a Cartesian product $\Omega = R \times S$. This is a complete description of how all uncertainty is resolved. Once r and s are known there is no further uncertainty.

The space, X, of outcomes is a connected subset of $\mathbb{R}$. An act is function, $a : \Omega \to X$. (For simplicity, we assume that the pay-offs of acts are expressed in utility terms.) The set of all acts is denoted by $A(\Omega)$. The decision maker has preferences represented by a binary relation, $\succsim$, on $A(\Omega)$.

A constant act is one which assigns the same outcome $x \in X$, to every pair $\langle r, s \rangle$ in Ω. Define

$$A(S) = \{a \in A(\Omega) : \forall \hat{r}, \tilde{r} \in R, \forall s \in S,\ a(\hat{r}, s) = a(\tilde{r}, s)\}$$

and

$$A(R) = \{a \in A(\Omega) : \forall r \in R, \forall \hat{s}, \tilde{s} \in S,\ a(r, \hat{s}) = a(r, \tilde{s})\}.$$

[4]For a finite state space this is without loss of generality, since null states can be deleted without affecting the analysis.

Acts are in $A(S)$ (respectively, $A(R)$) if and only if their outcome does not depend upon the randomizing device (respectively, the element in S that obtains). We shall denote generic elements of $A(S)$ by f, g, h (alluding to Savage acts that involve only the resolution of purely subjective uncertainty) and we shall denote generic elements of $A(R)$ by ℓ, ℓ', ℓ'' (alluding to lotteries with 'objective' probabilities).

To aid our exposition, for a generic product-space $D \times D'$, with generic element (d, d'), we consider the following definitions and restrictions based on $D = R$ (respectively, S) and $D' = S$ (respectively, R).[5]

First we assume for any non-empty subset $C \subseteq D$ the decision maker has a conditional preference $\succsim_E$ defined on $A(\Omega)$. Next, we note that the decision maker may randomize between acts in $A(D')$ as follows.

Definition 2.1 (**D-randomization**) *Suppose a, $b \in A(D')$ and $C \subsetneq D$, $C \neq \varnothing$, define an act $a^C b$ by $a^C b(d, d') = a(d, d')$ if $d \in C$, and $a^C b(d, d') = b(d, d')$ if $d' \notin C$.*

For the case $D = R$, the act $f^E g$ is a randomization between the purely subjectively uncertain acts f and g. This is the type of randomization considered by Raiffa (1961). If the resolution of R were to be made known to the decision-maker before the resolution of S then $f^E g$ could be viewed as a contingent plan in which the act f is chosen if the randomizing device selects an element of E and the act g is chosen otherwise. Similarly, for the case $D = S$, the act $\ell^F \ell'$ is a (possibly ambiguous) randomization between the lotteries ℓ and ℓ'. If the resolution of S were to be made known to the decision-maker before the resolution of R then $\ell^F \ell'$ could be viewed as a contingent plan in which the lottery ℓ is chosen if the event F obtains and the lottery ℓ' is chosen otherwise.[6]

Attitudes towards randomization are defined as follows.

[5] For the case $D = S$ (and hence $D' = R$), with slight abuse of notation we shall identify any act $\hat{a} \in D \times D'$ with the act $a \in A(\Omega)$, where $a(r, s) = \hat{a}(s, r)$, for all $(r, s) \in \Omega$.

[6] In a recent paper, Riedel and Sass (2012) study games where players can choose ambiguous randomization devices.

Definition 2.2 (Attitudes toward D-randomization) *An individual has a strict preference for (respectively, strict aversion, indifference to) $D-$randomization if for all non-empty $C \subsetneq D$, and all $a,\ b \in A(D')$ such that $a \sim b$, $a^C b \succ a$, (respectively, $a \succ a^C b$, $a^C b \sim a$).*[7]

Our main assumptions follow.

Assumption 2.1 (D-consequentialism) *For any non-empty $C \subsetneq D$, and any $a, b \in A(D')$, $a \sim_C a^C b$.*

This assumption requires that the conditional preference $\succsim_C$ only depends on the outcomes in $C \times D'$. Hence D-consequentialism may be viewed as a weak version of consequentialism in the sense that it only applies to events in D. If $D = R$, then $R-$consequentialism concerns only events defined by the randomizing device. Thus it does not exclude violations of consequentialism due to ambiguity.

Next is our assumption, which captures a notion that the D-space describes the realization of a randomizing device that is independent of the preferences over $A(D')$.

Assumption 2.2 (D-independent randomizations) *For any non-empty event $C \subsetneq D$, and any $a, b \in A(D')$, $a \succsim b \Leftrightarrow a \succsim_C b$.*

Consider first the case of R-independent randomization. In our opinion, the key features of an independent randomizing device are that its output conveys no useful information about the likelihood of the states and that the decision maker does not care directly about the outcome of the randomizing device. We believe that for two acts which only depend on the S states, preferences conditional on events defined by the randomizing device should coincide with the unconditional preference. It is logically possible that preferences might depend directly on the outcome of the randomizing device

[7] Recall that we are assuming that neither R nor S contains any null states.

even though it is independent of the state-uncertainty. For instance an individual may be more willing to accept an ambiguous risk if his/her lucky number comes up. The R-independent randomizations assumption rules out such preferences. We believe the S-independent randomization assumption is less compelling. However we have stated it here to allow comparisons.

Assumption 2.3 (D-dynamic consistency) *For any non-empty $C \subsetneq D$ and any pair of acts $a, b \in A(\Omega)$: $a \succsim_C b$ and $a \succsim_{C^c} b$ implies $a \succsim b$.*

As usual D-dynamic consistency says that an individual who conditionally expresses a preference between two acts whether or not the randomization based on D yields an outcome in the event C, should also unconditionally express that preference. This is a weak dynamic consistency axiom since it only requires consistency *after observing an event* in D.[8] Moreover, we believe the R-dynamic consistency axiom is compelling for ex-ante randomizations. It is not as strongly motivated for ex-post randomizations and we do not wish to apply it in that context.

We have introduced the assumptions in such generality so that we can distinguish

- randomizing devices for which the probability distribution may be
  - known (R-randomization) or
  - unknown (S-randomization) and
- sequences of information revelation where the outcome of the randomization is revealed
  - before states in S become known (ex-ante randomization) or
  - after these states become known (ex-post randomization).

[8] In this sense it may also be viewed as a weaker form of the "coherence" condition of Skiadas (1997), expression (6) p.353.

2.1 Ex-ante randomization

Our main motivation is to study ex-ante randomizations with known probabilities, which is the case suggested in Raiffa (1961). The dynamic information structure is clearly spelled out in Raiffa's thought experiment.

Our first result says that it cannot be the case that the decision-maker always prefers randomizations. If f and g are indifferent and some randomizations are preferred to the pure acts there must be other randomizations which are not superior to the pure acts.[9]

Proposition 2.1 *Let $\{\succsim, \succsim_E : E \subseteq R, E \neq \varnothing\}$ be a family of conditional preference orders that satisfies R-consequentialism (2.2) and R-dynamic consistency. Consider a given non-empty event $E \subsetneqq R$ and two acts $f, g \in A(S)$ such that $f \sim g$, if $f^E g \succ f$ then $f \succsim g^E f$.*

Proof. Notice first that R-consequentialism implies $f \sim_E f^E g$, $g \sim_E g^E f$, $f \sim_{E^c} g^E f$ and $g \sim_{E^c} f^E g$. Now suppose $f \sim g$. By R-dynamic consistency $f^E g \succ f$ implies $g \succ_{E^c} f$ (otherwise, $f \sim_E f$ and $f \succsim_{E^c} g$ would imply $f \succsim f^E g$, which is a contradiction). Applying $R-$dynamic consistency again, $g \succ_{E^c} f \sim_{E^c} g^E f$ and $g \succsim_E g \sim_E g^E f$ implies $g \succsim g^E f$. Hence $f \succsim g^E f$. ■

Proposition 2.1 does not make use of R-independent randomizations, (2.2). If we add this to our previous assumptions we can deduce that preferences must exhibit indifference to randomization.

Proposition 2.2 *Let $\{\succsim, \succsim_E : E \subseteq R, E \neq \varnothing\}$ be a family of conditional preference orders that satisfies R-consequentialism (2.2), R-independent randomizations (2.2) and R-dynamic consistency (2.3). Then the unconditional preference order $\succsim$ must be indifferent to R-randomization.*

[9] One may find this result less compelling in a normative sense than a strict preference which violates dynamic consistency. If the set of outcomes is a continuum and if preferences satisfy an appropriate continuity property, however, we can construct a violation of dynamic consistency which only involves strict preferences. The argument in the introduction provides an outline of the proof.

Proof. Fix a pair of acts $f, g \in A(S)$ such that $f \sim g$ and a non-empty $E \subsetneqq R$. Then $f \sim g \Rightarrow f^E g \sim_E f$, by R-consequentialism. Also $f \sim g \Rightarrow f \sim_{E^c} g$, by R-independent randomization. Now $f \sim_{E^c} g \Rightarrow f \sim_{E^c} f^E g$, by R-consequentialism. But if $f^E g \sim_E f$ and $f^E g \sim_{E^c} f$ then by R-dynamic consistency $f^E g \sim f$. Hence we have shown $f \sim g$ implies $f^E g \sim f$ for all non-empty $E \subsetneqq R$, and thus $\succsim$ is indifferent to randomization. ■

Proposition 2.2 does not assume beliefs over the S-space are ambiguous. Hence it also applies to some non-expected utility preferences which are probabilistically sophisticated in the sense of Machina and Schmeidler (1992). Its proof also does not use that the fact that beliefs about the randomizing device are additive. This suggests that even if an ambiguous randomizing device is available, the decision maker would not wish to use it.[10]

Indeed, by reversing the roles of R and S in the statement and proof of Proposition 2.2 we obtain the following corollary.

Corollary 2.1 *Let $\{\succsim, \succsim_F : F \subseteq S, F \neq \varnothing\}$ be a family of conditional preference orders that satisfies S-consequentialism, S-independent randomizations and S-dynamic consistency. Then the unconditional preference order $\succsim$ must be indifferent to S-randomization.*

2.2 Ex-post Randomization

In the AA framework, acts result in probability distributions over outcomes. It has been pointed out by Kreps (1988) and others that, in this approach, randomizations over acts correspond to ex-post randomizations over the outcome probability distributions. While Anscombe and Aumann (1963) introduce this interpretation of a randomization over acts as an explicit assumption,[11] most of the subsequent literature on decision making

[10] Bade (2011) makes a related point. She shows that in a strategic setting an ambiguous randomizing device would not help a player in a game. In her model the equilibria with ambiguous randomizations coincide with conventional Nash equilibria.

[11] Anscombe and Aumann (1963) p. 201, Assumption 2 (Reversal of order of compound lotteries).

under uncertainty takes this interpretation as an unquestioned characteristic of the AA approach itself.

Indeed, if one takes a randomization over AA acts as a state-wise randomization over the outcome lotteries, as almost all of the literature does, then it is possible to show that such ex-post randomizations may be preferred to the ambiguous act itself by ambiguity-averse decision makers. In particular, a decision maker who prefers to bet on the unambiguous urn as described in Ellsberg's two-colour urn (Ellsberg (1961)) will prefer an ex-post randomization.

Consider a family of conditional preferences where the conditioning is on subsets of the state space S and the unconditional preferences exhibit the following notion of ambiguity aversion.

Definition 2.3 (Ellsberg ambiguity-averse) *A decision maker with preferences* $\succsim$ *on* $A(\Omega)$ *is* Ellsberg ambiguity-averse *if, for some* $E \subset R$*,* $F \subset S$*, and some pair of outcomes* $x \succ y$*,* $x^E y \sim y^E x \succ x^F y \sim y^F x$*.*

Notice that the indifferences $x^E y \sim y^E x$ and $x^F y \sim y^F x$ for some $x \succ y$, suggests the decision maker views both the event $E \times S$ and the event $R \times F$ as equally likely as its complement, respectively. If the decision-maker were probabilistically sophisticated in the sense of Machina and Schmeidler (1992) then this would require an indifference between the lottery $x^E y$ and the act $x^F y$. Instead the lack of indifference indicates a strict preference of the decision maker to bet on lotteries rather than acts.[12]

As the next proposition shows, in conjunction with S-consequentialism and S-dynamic consistency, such an individual will exhibit a strict preference for some R-randomization over some pairs of acts in $A(S)$.

[12] This may be viewed as a special case of 'issue preference' (as in Ergin and Gul (2009)) or 'source dependence' (as in Chew and Sagi (2008))

Proposition 2.3 *Let* $\left\{\succsim, \succsim_{\hat{F}} : \hat{F} \subseteq S,\ \hat{F} \neq \varnothing\right\}$ *be a family of conditional preference orders that satisfies* S*-consequentialism and* S*-dynamic consistency. If, in addition, the unconditional preferences* $\succsim$ *are Ellsberg ambiguity-averse then there exists a non-empty event* $E \subset R$*, and a pair of acts* f *and* g *in* $A(S)$*, such that* $f^E g \succ f \sim g$*. That is, the unconditional preferences exhibit a strict preference for some* R*-randomization over some pair of acts.*

Proof. Since the agent is Ellsberg ambiguity-averse there exists a subset $E \subset R$, $F \subset S$, and some pair of outcomes $x \succ y$, such that $x^E y \sim y^E x \succ x^F y \sim y^F x$. Set $f := x^F y$, $g := y^F x$, $\ell := x^E y$ and $\hat{\ell} := y^E x$. By construction $\ell \sim \hat{\ell} \succ f \sim g$. Consider now the randomization $f^E g$. By S-consequentialism,

$$\begin{aligned} f^E g &= \left(x^F y\right)^E \left(y^F x\right) = \left(x^E y\right)^F \left(y^E x\right) \\ &= \ell^F \hat{\ell} \sim_F \ell, \\ \text{and } f^E g &= \ell^F \hat{\ell} \sim_{F^c} \hat{\ell} \sim \ell. \end{aligned}$$

Hence by S-dynamic consistency, $f^E g \succsim \ell \succ f \sim g$. ■

Notice if, in addition to the hypotheses of Proposition 2.3, a decision maker's family of conditional preferences $\left\{\succsim, \succsim_{\hat{F}} : \hat{F} \subseteq S,\ \hat{F} \neq \varnothing\right\}$ also satisfied S-independent randomizations then it follows from Proposition 2.3 and Corollary 2.1 that her unconditional preferences must both be indifferent to S-randomizations as well as exhibiting a strict preference for some R-randomization for some pair of acts.

It is an empirical question whether Ellsberg ambiguity-averse people will actually prefer ex-post randomizations. In a recent experimental study, Eichberger, Oechssler, and Schnedler (2012) find evidence that subjects showing ambiguity aversion in the standard Ellsberg two-urn experiment do not prefer an ex-post randomization.

3 Paradoxes and Two Types of Randomization

In recent publications, Machina (2009), Baillon, L'Haridon, and Placido (2011), Epstein (2010), and Klibanoff, Marinacci, and Mukerji (2012) have appealed to intuition in order to justify preference rankings over uncertain acts and their randomizations that cannot be represented by common non-expected utility functionals. All these non-expected utility representations are based upon the AA framework which treats randomizations over acts as ex-post state-wise randomizations over the associated outcome probability distributions. We will argue that the intuition advanced to support the incompatible preferences are based on the ex-ante randomisation interpretation.

3.1 Paradoxes for the Smooth Ambiguity Model

Epstein (2010) provides examples of acts for which the intuitive ranking is incompatible with the representation by the smooth model proposed by Klibanoff, Marinacci, and Mukerji (2005). In terms of our notation developed in section 2, his example may be expressed as follows.

Example 3.1 (Epstein (2010)) *You are given two urns, numbered* 1 *and* 2*, each containing* 50 *balls that are either black or white. You are told also that the two urns are generated independently, for example, they are set up by administrators from opposite sides of the planet who have never been in contact with one another. One ball will be drawn from each urn. Thus, we take* $\Omega = R \times S$*, where* $R = \{h, t\}$ *and* $S = \{b_1, w_1\} \times \{b_2, w_2\}$*, and for which* h *(respectively,* t*) corresponds to a 'fair-coin' coming up heads (respectively, tails), and* $(s_1, s_2) \in \{b_1, w_1\} \times \{b_2, w_2\}$*, corresponds to the colour of the ball drawn from the first (respectively, second) urn being* s_1 *(respectively,* s_2*). Notice that for any pair of outcomes* $x > y$*, the acts* $x^{\{h\}}y$ *and* $y^{\{h\}}x$ *(in* $A(R)$*) both correspond to an equal* *probability lottery over this pair of outcomes. Consider the following acts:*

Bets for Experiment 2

	$(s_1, s_2) \in S$			
	(b_1, b_2)	(b_1, w_2)	(w_1, b_2)	(w_1, w_2)
f_1	x	x	y	y
f_2	x	y	x	y
$f_1^{\{h\}} f_2$	x	$x^{\{h\}} y$	$y^{\{h\}} x$	y
g	y	y	x	x
a_1	$x^{\{h\}} y$	$x^{\{h\}} y$	$y^{\{h\}} x$	$y^{\{h\}} x$
a_2	$x^{\{h\}} y$	y	x	$y^{\{h\}} x$

Epstein (2010) argues for the preference ranking $f_1 \sim f_2 \sim f_1^{\{h\}} f_2$ and $a_1 \succ a_2$ and shows that this ranking of acts cannot be represented by the smooth model, since $a_1 = f_1^{\{h\}} g$ and $a_2 = f_2^{\{h\}} g$. In order to support this ranking Epstein (2010) writes:[13]

> "Symmetry suggests indifference between f_1 and f_2. If it is believed that the compositions of the two urns are unrelated, then f_1 and f_2 do not hedge one another. If, as in the multiple-priors model, hedging ambiguity is the only motivation for randomizing, then we are led to the rankings $f_1 \sim f_2 \sim f_1^{\{h\}} f_2$. Ambiguity aversion suggests $a_1 \succ a_2$."

In their reply, Klibanoff, Marinacci, and Mukerji (2012) do not question the result of Epstein, but challenge his intuition:[14]

> "Epstein argued that $f_1^{\{h\}} f_2 \sim f_1 \sim f_2$ and $a_1 \succ a_2$ are natural for a strictly ambiguity averse individual. .... We agree with the intuition for $[a_1 \succ a_2]$,

[13] In the quotation, the notation of events and acts has been adapted to the one used in this paper.
[14] As was the case in the previous quotation, we have adapted the notation of events and acts to the one used in this paper.

but disagree that $f_1^{\{h\}}f_2 \sim f_1 \sim f_2$ is natural for an ambiguity averse individual and think there is good reason to expect $f_1^{\{h\}}f_2 \succ f_1 \sim f_2$. ... The evaluation of $f_1^{\{h\}}f_2$ depends on the colour compositions of both urns, but has half the exposure to the uncertainty about the ratio in each urn compared to f_1 and f_2. Recall that the determination of the two urn compositions is viewed as independent. The act $f_1^{\{h\}}f_2$ thus diversifies the individual's exposure across the urns: it provides a hedging of the two independent ambiguities in the same sense as diversifying across bets on independent risks provides a hedging of the risks. To an individual who is averse to ambiguity (i.e., to subjective uncertainty about relative likelihoods), such diversification is naturally valuable."

Epstein's argument that the indifference between f_1 and f_2 cannot be improved upon by a randomization because "the compositions of the two urns are unrelated" clearly corresponds to the ex-ante view advanced above. In contrast, the hedging argument of Klibanoff, Marinacci, and Mukerji refers explicitly to the (ex-post) equivalence of the 'lottery' outcomes $x^{\{h\}}y$ and $y^{\{h\}}x$ in states (b, w) and (w, b), respectively, achieved by the 'randomization' $f_1^{\{h\}}f_2$.

3.2 The Reflection Paradox

Machina (2009) discusses several examples in order to argue that a certain non-separability features of acts cannot be represented by CEU preferences. Baillon, L'Haridon, and Placido (2011) extend this line of argument to other non-expected utility representations. Reconsidering one of Machina's examples, the "Reflection Example", we argue that the intuition for the preference orders that cannot be represented rests on the ex-ante view. Since almost all non-expected utility representations rely on the AA framework with its ex-post **interpretation of randomizations,** they are all vulnerable to preference rankings built on the ex-ante view.

Example 3.2 (Machina (2009)) *Consider the* Ellsberg two-urn problem *where it is known that the first urn contains 50 black balls and 50 white balls and the second urn has 100 balls with an unknown proportion of black and white balls. Thus, we take* $\Omega = R \times S$*, where* $R = \{b_1, w_1\}$ *and* $S = \{b_2, w_2\}$*, and for which* b_i *(respectively,* w_i*),* $i = 1, 2$*, corresponds to the colour of the ball drawn from urn* i *being black (respectively, white). Table I defines four acts.*

Table I

	$(r,s) \in R \times S$			
	(b_1, b_2)	(b_1, w_2)	(w_1, b_2)	(w_1, w_2)
a_1	\$4000	\$8000	\$4000	\$0
a_2	\$4000	\$4000	\$8000	\$0
a_3	\$0	\$8000	\$4000	\$4000
a_4	\$0	\$4000	\$8000	\$4000

Machina finds the preference order

$$a_2 \sim a_3 \succ a_1 \sim a_4$$

intuitive based on the following argument (Machina (2009), 389-390)[15]:

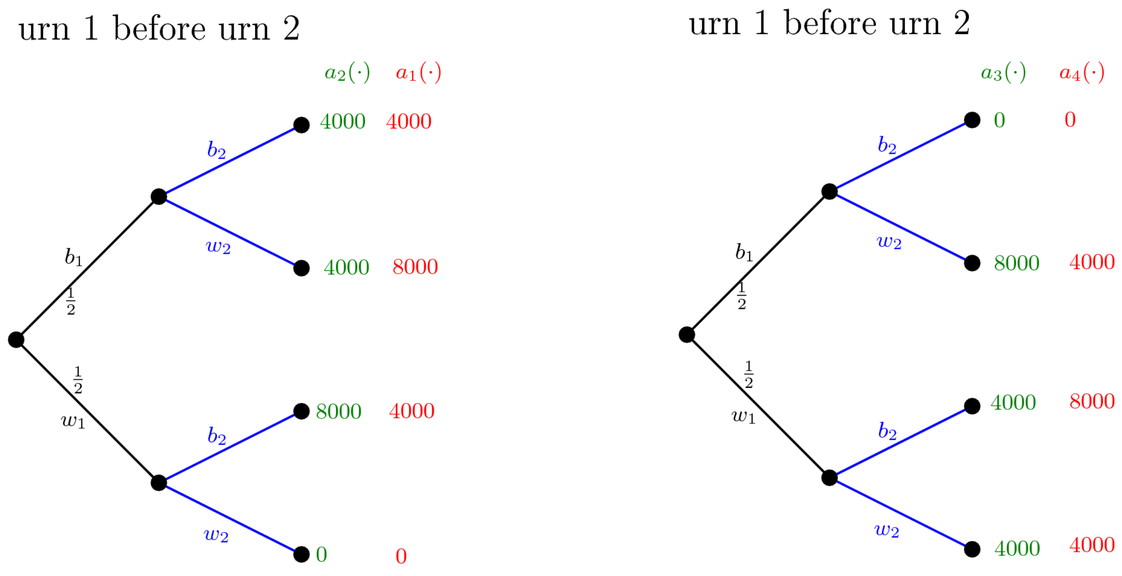
> "Consider a choice between a_1 and a_2 ... It is not clear which of these acts is more ambiguous: a_2 has a payoff difference of \$8,000 riding on the subjective sub-partition $\{(w_1, b_2), (w_1, w_2)\}$, whereas a_1 divides this stake, with \$4,000

[15] In the quotation, the notation of events and acts has been adapted to the one used in this paper.

riding on each of the sub-partitions $\{(b_1, b_2), (b_1, w_2)\}$ and $\{(w_1, b_2), (w_1, w_2)\}$. Thus, a_1 gives a 100 percent chance of $\$4,000$ riding on the unknown composition of the urn, whereas a_2 gives a 50 percent chance of $\$8,000$ riding on this subjective uncertainty.

Say the individual does have a strict preference, $a_1 \succ a_2$. ... But as seen in Table $[I]$, a_4 is an informationally symmetric left-right reflection of a_1, and a_3 is a left-right reflection of a_2. Surely, anyone with the ranking $a_1 \succ a_2$ should have the 'reflected' ranking $a_3 \prec a_4$. ... Call this urn and bets the *reflection example*."

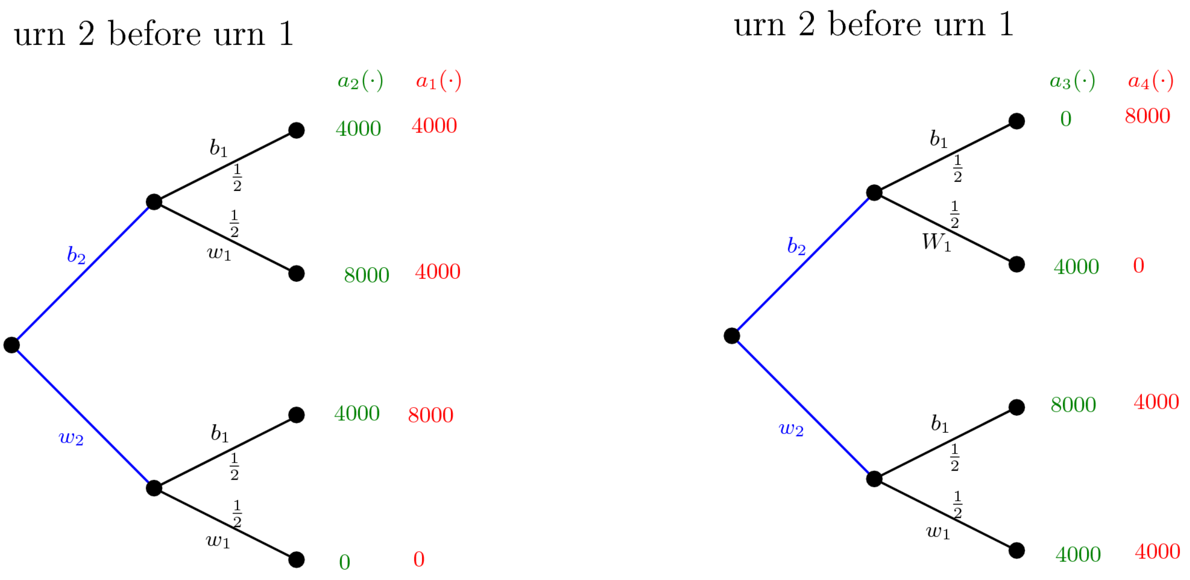
This intuition can be illustrated by the following diagram and table which considers the draw from urn 1 as an ex-ante randomization.



$r \in \{b_1, w_1\}$		
	b_1	w_1
a_1	$4000^{\{b_2\}}8000$	$4000^{\{b_2\}}0$
a_2	4000	$8000^{\{b_2\}}0$
a_3	$0^{\{b_2\}}8000$	4000
a_4	$0^{\{b_2\}}4000$	$8000^{\{b_2\}}4000$

Clearly, a_2 and a_3 are equivalent if one views a_2 (respectively, a_3) as a fifty-fifty randomization over the constant act 4000 and the act $8000^{\{b_2\}}0$ (respectively, $0^{\{b_2\}}8000$). Similarly, from this perspective, a_1 (respectively, a_4) is a fifty-fifty randomization over the pair of acts $4000^{\{b_2\}}8000$ and $4000^{\{b_2\}}0$ (respectively, $8000^{\{b_2\}}4000$ and $0^{\{b_2\}}4000$). On the other hand, $a_3 \succ a_4$ appears plausible given the guaranteed pay-out of 4000 in the event the ball drawn from the first urn is white. Hence, $a_2 \sim a_3 \succ a_1 \sim a_4$ seems justified from the ex-ante perspective.

One may, however, also take the ex-post perspective. By reversing the order of the draws we obtain the following picture.



‘AA acts’

	$s \in \{b_2, w_2\}$	
	b_2	w_2
a_1	4000	$8000^{\{b_1\}}0$
a_2	$4000^{\{b_1\}}8000$	$4000^{\{b_1\}}0$
a_3	$0^{\{b_1\}}4000$	$8000^{\{b_1\}}4000$
a_4	$0^{\{b_1\}}8000$	4000

In this ex-post view, the reverse ordering

$$a_2 \sim a_3 \prec a_1 \sim a_4,$$

seems plausible. Now assume that the pay-offs are not utilities but are cash amounts. Let U denote the decision-maker's utility of wealth, where, without loss of generality, we set $U(8000) := 1 > u =: U(4000) > 0 =: U(0)$. Then the ex-post expected utilities are as follows:

expected utilities

	$s \in \{b_2, w_2\}$	
	b_2	w_2
$U \circ a_1$	u	$\frac{1}{2}$
$U \circ a_2$	$\frac{1}{2}(u+1)$	$\frac{1}{2}u$
$U \circ a_3$	$\frac{1}{2}u$	$\frac{1}{2}(1+u)$
$U \circ a_4$	$\frac{1}{2}$	u

Comparing $U \circ a_2$ and $U \circ a_1$ we see that the highest utility is higher and the lowest utility is lower with $U \circ a_2$. Indeed if the probability of b_2 is known to be $\frac{1}{2}$, $U \circ a_2$ and $U \circ a_3$ are mean-*expected utility* preserving spreads of $U \circ a_1$ and $U \circ a_4$, respectively.

4 Conclusion

This paper studies the Raiffa argument for randomization as a response to ambiguity. It shows that preferences which are not indifferent to randomization must be dynamically inconsistent. If one views dynamic consistency as a fundamental normative principle, then these results suggest there is no normative theory of ambiguity aversion in which there is a strict preference for randomization.

Previously two of us studied randomization in the context of the convex capacity model, Eichberger and Kelsey (1996). Subsequently we studied dynamic consistency in the same context, Eichberger, Grant, and Kelsey (2005). Recently for CEU preferences which are not necessarily ambiguity-averse we found conditions for ambiguity-attitude to be unchanged after updating, Eichberger, Grant, and Kelsey (2012). Similar conditions were found to be sufficient in all three cases. Propositions 2.1 and 2.2 can be seen as extending the earlier analysis to a model free framework.

From the descriptive point of view there is little evidence that subjects in experiments are dynamically consistent. Thus there is no logical reason why one cannot assume strict preference for randomization in a descriptive theory. However the experimental evidence does not suggest that individuals express a strict preference for randomization, Dominiak and Schnedler (2011).

The preference for randomization argument is most plausible when applied to problems of balls and urns. In an experiment like the 2-ball Ellsberg urn it is reasonably clear that betting on red is a complementary act to betting on black. In real ambiguous decisions there are often no easily available acts which pay-off in the complementary events. Consider, for instance, decisions such as whether it is worth paying a large amount of money to protect against an uncertain environmental threat or whether to invade a rogue state which may or may not have weapons of mass destruction. In such cases randomization is not obviously attractive. Ball and urn experiments capture some aspects of reality. However they also leave important things out.

In the convex capacity model it is not possible to have a strict preference for randomization. In more general models of ambiguity, such as multiple priors, one may or may not have a strict preference for randomization. Both preferences are possible, and which one uses is a modelling choice. We anticipate that some researchers will continue to assume a strict preference for randomization. The price of doing so is dynamic inconsistency.

In this paper we have studied the relationship between preference for randomization and dynamic consistency. We have found that a crucial issue is whether the outcome of the randomizing device is revealed before or after the state uncertainty is resolved.

In the present paper we use a Savage model in which the order of resolution of the subjective uncertainty and the randomizing device is not explicitly modelled. In such a framework one can interpret the randomization as being either ex-ante or ex-post by adopting appropriate axioms. However it is not possible to model ex-post and ex-ante randomization at the same time. A topic for future research is to develop a model in which one could study both kind of randomization simultaneously. This theory would extend earlier work by Saito (2012) which presents a model in which decision-makers could have different attitudes to the two kinds of randomization simultaneously.

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