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A two-parameter model of dispersion aversion

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Abstract

The idea of representing choice under uncertainty as a trade-off between mean returns and some measure of risk or uncertainty is fundamental to the analysis of investment decisions. In this paper, we show that preferences can be characterized in this way, even in the absence of objective probabilities. We develop a model of uncertainty averse preferences that is based on a mean and a measure of the dispersion of the state-wise utility of an act. The dispersion measure exhibits positive linear homogeneity, sub-additivity, translation invariance and complementary symmetry. Since preferences are only weakly separable in terms of these two summary statistics, the uncertainty premium need not be constant. We generalize the concept of decreasing absolute risk aversion. Further we derive two-fund separation and asset pricing results analogous to those that hold for the standard CAPM.

JEL classification: D81; G11; G12

Keywords: Uncertainty aversion; Mean utility; Dispersion of utility; Weak-separability; Two-fund separation; CAPM excess return formula

1. Introduction: dispersion versus return

Ever since the pioneering work of Markowitz [18] and Tobin [24], the idea of representing investment decisions in terms of a trade-off between risk (often characterized by some measure of the dispersion or variation of the return) and expected return has played a prominent role

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in finance theory. The mean-variance analysis presented by Markowitz and Tobin formed the basis of the Capital Asset Pricing Model (CAPM) (Sharpe [22], Lintner [14]) which remains the main workhorse of financial analysis. However, mean-variance analysis has been subject to a wide range of criticisms. The first criticism came from proponents of expected utility theory (EUT), who observed that mean-variance analysis was consistent with EUT only for the special (and unattractive) case of a quadratic utility function.[1] If the EUT hypothesis is abandoned, it is possible to consider more general mean-variance preferences, but these are typically ad hoc functional forms, lacking the axiomatic foundations that characterize EUT.

A more recent set of criticisms relates to the choice of the variance or equivalently, the standard deviation as the measure of risk. While the standard deviation has appealing qualities,[2] a large body of evidence suggests that the return distributions for many assets are 'fat-tailed' having excess kurtosis relative to the normal. This suggests the need either to take higher moments into account, which substantially complicates the analysis, or to use measures of riskiness other than the standard deviation.

More fundamental criticisms arise from the work of Ellsberg [4]. Mean-variance analysis typically treats probabilities as if they are objectively known, or at least as if they can be derived from observed preferences as in Savage [21]. But there is ample evidence to suggest that many decision-makers do not display preferences consistent with well-defined subjective probabilities (probabilistic sophistication in the terminology of Machina and Schmeidler [17]). In particular, preferences may display source dependence as in Chew and Sagi [3] or Ergin and Gul [6]. Decision-makers may prefer either side of a symmetric bet that is well-understood, such as a coin toss, over either side of an apparently symmetric bet on an unfamiliar event, such as up or down daily movements in temperature in an unfamiliar city.[3]

In this paper, we address all of these issues. We provide a rigorous foundation for preferences characterized by two arguments, a mean value and a dispersion parameter. The properties of the dispersion parameter generalize those of the standard deviation and are satisfied (modulo an appropriate normalization in some cases) by many of the commonly used measures of dispersion in the statistical literature. Our approach, however, encompasses choice under risk (known objective probabilities), choice under uncertainty (subjective probabilities as in Savage [21]) and choice under ambiguity (where different 'sources' of uncertainty need not be treated symmetrically).

2. Background

In Grant and Polak [12] two of the co-authors of the present paper examine the family of mean-dispersion preferences that admit a representation that takes the form of a 'mean' minus a 'dispersion measure' of the state-contingent utility vector associated with an act. In particular, for these preferences, one can show that there exist an affine utility function U over consequences, a probability weighting vector π on the states and a function ρ over state-utility vectors such that the preferences over acts are represented by the functional

$$V(f) = \mathrm{E}_{\pi}(U \circ f) - \rho(U \circ f), \tag{1}$$

[1] For example, expected utility with quadratic utility implies that risk preferences exhibit increasing absolute risk aversion.

[2] These are most evident for the case of normal distributions, which are fully characterized by the mean and standard deviation.

[3] Such a source dependence preference may be invoked to explain the phenomenon of home market bias in investment decisions (French and Poterba [9]).

where $U \circ f$ is the utility vector given by $(U \circ f)_s := U(f(s))$ and $\mathrm{E}_\pi(u) := \sum_s \pi_s u_s$, for each utility vector u. Moreover, $\rho(0) = 0$ and ρ exhibits translation invariance in the sense that, letting e denote the constant state-utility vector $(1, \dots, 1)$, we have $\rho(u + \delta e) = \rho(u)$, for all δ.

For the case where $\rho(u) \geqslant 0$, we can view $\rho(u)$ as a measure of dispersion of the utility vector u. The interpretation is that the agent with these preferences dislikes dispersion. More specifically, for each act f, let x_f be a constant act such that $x_f \sim f$. Then, the measure of dispersion $\rho(U \circ f)$ associated with the act f is given by $\mathrm{E}_\pi(U \circ f) - U(x_f)$: it is the reduction in expected utility the agent would be willing to accept in return for removing all the state-contingent utility uncertainty associated with the act. Drawing an analogy from choice under risk, we can think of $\mathrm{E}_\pi(U \circ f)$ as corresponding to a certainty equivalent and of $\rho(U \circ f)$ as corresponding to an absolute risk premium. Thus, $\rho(U \circ f)$ is an 'absolute uncertainty premium'.[4] Since ρ exhibits translation invariance and V is linear in ρ, mean-dispersion preferences exhibit the property of *constant absolute uncertainty aversion*.

Siniscalchi [23] characterizes an important special case of such preferences which he calls vector expected utility preferences. In his model, however, ρ satisfies 'complementary symmetry'. Essentially complementary symmetry entails that for any utility vector u we require $\rho(u) = \rho(-u)$. Grant and Polak [12] show that mean-dispersion preferences include the variational preferences of Maccheroni et al. [15] (and thus also the multiple prior model of Gilboa and Schmeidler [11] and the multiplier preferences of Hansen and Sargent [13]).

The starting point for this paper is the observation that a constant uncertainty premium is a restrictive assumption. How plausible we find this restriction may depend on the stories we use to interpret these models. For example, one could interpret a multiple-prior set as simply reflecting the set of probabilities over states of the world that the agent perceives as possible. There is no reason for this perceived set to change as the agent becomes better off, and so, in this interpretation, a constant uncertainty premium is perhaps quite plausible. But an alternative interpretation of mean dispersion preferences (even in the multiple-prior case) is that they reflect not just the agent's perceptions of ambiguity but also the agent's dislike of any perceived ambiguity.[5] Indeed, the term 'ambiguity averse' seems to suggest dislike rather than just perception. If we believe this dislike-of-ambiguity interpretation then it seems less plausible that uncertainty premiums should be constant.

For choices with monetary payoffs, expected utility theory can allow for flexibility in the trade-off between mean-payoff and the dispersion of payoffs at different levels of wealth. The concept of decreasing absolute risk aversion captures the intuition that as wealth increases, so will the willingness to accept a bet with a given mean and dispersion in monetary returns. But there are other settings in which the most natural one-dimensional 'pay-off' for the individual is the Bernoulli utility index itself, in which case the expected utility hypothesis necessarily entails a constant neutral attitude toward dispersion.

To motivate why we may wish to allow for both non-neutral and non-constant attitudes toward dispersion of state-contingent utilities, consider a patient deciding between treatments for an

[4] This 'uncertainty premium' is the amount that the agent is willing to pay to ensure that she obtains the same utility in every state. This premium corresponds to the notion of 'ambiguity' aversion in Ghirardato and Marinacci [10].

[5] For example, in one story used to interpret ε-perturbation models, the agent is thought of as perceiving a large set of possible probabilities (for example, the entire simplex) but then only putting weight ε on the 'worst probability' from this set with the remaining $(1 - \varepsilon)$ on a particular prior. In this case, the 'revealed-preference' multiple-prior set is obviously much smaller than the entire simplex. If we believe the story, it is not obvious that the ε (and hence the revealed multiple-prior set) should remain constant as the agent becomes better off.

acute health condition. There are two possible outcomes that may result from treatment, recovery or death.

First consider a choice between treatments A and B. Suppose that treatment A is the standard treatment that is well-understood, and for which there is an extensive body of medical evidence. From this evidence it has been determined that for this patient the probability of recovery from treatment A is equal to $1/3$.

Treatment B is a relatively new experimental treatment for which there is still much uncertainty about its effectiveness and its attendant risks for different types of individuals. In the event that it is suitable for this particular patient which, we suppose, corresponds to the event E obtaining, the probability of recovery is equal to $2/3$. In the event that it is unsuitable for this particular patient, however, the treatment is fatal, that is, the probability of recovery is zero!

Next consider the same patient facing a choice between treatments A' and B', where treatment A' like A is well-understood with the probability of recovery equal to $2/3$ and treatment B' for which the probability of recovery is equal to 1 if state E obtains, and is equal to $1/3$ otherwise. Notice that for every state, the difference between treatments A and A' (respectively, B and B') in the state-contingent 'utility' measured in units of probability of recovery is $1/3$.

We contend that it is entirely plausible for this patient to express a strict preference for treatment A over treatment B and a strict preference for treatment B' over treatment A'. Such a pattern clearly contradicts subjective expected utility theory, since the former strict preference implies that the patient's subjective probability that event E will obtain is less than one-half, while the latter strict preference implies that her subjective probability is greater than one-half. But such a preference pattern is entirely consistent with an individual who exhibits decreasing absolute uncertainty aversion.

So with this in mind, we develop a model that *allows* uncertainty premiums to vary as we change mean utility.

The new model maintains the tractable feature that preferences can be expressed in terms of two summary statistics: a mean and a measure of dispersion of the state utility vector. But instead of being additively separable in terms of the mean and the measure of dispersion as is the case with mean-dispersion preferences considered in Grant and Polak [12] and variational preferences characterized in Maccheroni et al. [15], the preferences we consider need only be weakly separable. To aid tractability in applications, however, the class of preferences we characterize admit a representation in which the dispersion measure exhibits positive linear homogeneity, sub-additivity, translation invariance and complementary symmetry.[6] As a consequence, mean-dispersion preferences and variational preferences overlap but are not nested.

We call such preferences 'invariant symmetric preferences'. They take the general form:

$$V(f) = \varphi\big(\mathrm{E}_\pi(U \circ f), \rho(U \circ f)\big),$$

where φ is increasing in its first argument, non-increasing in its second argument, and $\varphi(y, 0) = y$. Using this latter property yields the following obvious, but informative, decomposition:

$$\begin{aligned} V(f) &= \mathrm{E}_\pi(U \circ f) - \big[\mathrm{E}_\pi(U \circ f) - \varphi\big(\mathrm{E}_\pi(U \circ f), \rho(U \circ f)\big)\big] \\ &= \mathrm{E}_\pi(U \circ f) - \big[\varphi\big(\mathrm{E}_\pi(U \circ f), 0\big) - \varphi\big(\mathrm{E}_\pi(U \circ f), \rho(U \circ f)\big)\big]. \end{aligned}$$

[6] Together the first two properties imply convexity. The first three characterize the finite state space analog of Rockafellar et al.'s [20] *general deviation measure*. All four properties are ones that typically hold for well-known measures of dispersion in the statistics literature, such as standard deviation, mean absolute deviation and Gini's mean difference.

That is, we may interpret the difference $\varphi(\mu, 0) - \varphi(\mu, \rho)$ as the 'absolute uncertainty premium' (measured in 'utils') of an (and any) act with mean utility μ and measure of dispersion ρ.

Some examples may aid intuition. First, consider the choice of ρ. The standard deviation is the most commonly used measure of dispersion. As with the 'normal' distribution, the term 'standard deviation' indicates the canonical role of this measure of dispersion in classical statistics. This canonical role arises from the Central Limit Theorem, which ensures that, under weak conditions, the distribution of the sample mean converges to the normal with large numbers of observations. The canonical status of the normal distribution is closely linked to the various forms of the Law of Large Numbers which ensure that observed sample frequencies converge to objective probabilities, again with large numbers of observations.

The canonical status of the normal distribution is far less appealing under the conditions of ambiguity that characterize stock market investments. Even given large volumes of historical data, there is no objective basis for determining the distribution of future returns. Furthermore, the observed distribution of returns is typically not normal, but fat-tailed. Such fat-tailed distributions commonly display infinite variance.

In these circumstances, it may be more appropriate to consider distributions such as the class of symmetric stable distributions which includes the normal (Gaussian), Cauchy and Levy distributions. Any member of the family is characterized by three parameters one of which is a scale parameter, proportional to the standard deviation in the Gaussian case. For general stable distributions, the scale parameter is a natural dispersion measure exhibiting all the properties we listed above, such as translation invariance and symmetry. As shown by Fama and Roll [7], the inter-quartile range of a sample is a robust estimator of the scale parameter for a wide range of stable distributions.[7]

Second, consider the function φ. The usual mean-variance model is linear in the mean and the square of the standard deviation, thus exhibiting constant absolute risk aversion. Epstein [5] introduced a more general mean-variance model precisely to capture the idea of decreasing absolute risk aversion. His mean-variance functionals are weakly separable in the mean and the standard deviation, just as in expression (1). Thus, we can think of Epstein's model as an example of our more general invariant symmetric preferences.

The closest analog of this model in the context of risk preferences is the invariant risk preferences introduced by Quiggin and Chambers [19]. Indeed, we chose the name 'invariant symmetric' since we view this class of preferences as an analog of Quiggin and Chambers' invariant risk preferences for the setting of subjective uncertainty, albeit with a symmetric (but *not necessarily* [second-order] probabilistically sophisticated) dispersion measure playing the role of their risk measure. Furthermore, since we are in a setting of subjective uncertainty, in our model the probabilities over the states are not given exogenously but rather are derived as part of the representation from purely behavioral properties of the underlying preferences.

In what follows, we provide an axiomatization of the invariant symmetric preferences model. Our axioms utilize a key axiom from Siniscalchi's [23] axiomatization of vector expected utility preferences. Like Maccheroni et al. [15] we weaken Gilboa and Schmeidler's [11] constant independence axiom which itself was a weakening of the Anscombe–Aumann [1] independence axiom. Thus, the standard subjective expected utility model is nested in our axiomatization.

[7] If the data generating process is Cauchy, then the inter-quartile range of the population distribution is equal to the scale parameter and so the sample inter-quartile range is an unbiased estimator.

Section 4 introduces the main axioms and main representation theorem for our more general mean-dispersion preferences. We show in Sections 5 and 6 how the model can accommodate non-constant uncertainty aversion. In Section 7, we show in a finance setting that, if the investors' preferences admit an invariant symmetric representation with a common baseline probability and a linear Bernoulli utility index, then a two-fund separation result holds and, furthermore, one can derive a CAPM style asset pricing formula.

3. Invariant symmetric preferences

Our set-up is similar to Maccheroni et al.'s [15] except we take the state space S to be a finite set $\{s_1, \dots, s_n\}$ denoting the possible states of nature that may obtain. Let X be the set of consequences. An act is a function $f: S \to X$. With slight abuse of notation, any x in X will also denote the constant act that yields x in every state. Let $\mathcal{F}$ denote the set of acts and, continuing our abuse of notation, X shall also denote the set of constant acts. In addition we shall assume that X is a convex subset of a vector space. For example, in Fishburn's [8] rendition of the Anscombe–Aumann [1] setting, X is taken to be the set of all lotteries on a set of final prizes. Alternatively, in finance applications, X is often taken to be a subset of the positive reals. This means both the sets X and $\mathcal{F}$ are mixture spaces. In particular, for any pair of acts f and g in $\mathcal{F}$, and any α in $(0, 1)$, we take $\alpha f + (1 - \alpha)g$ to be the act $h \in \mathcal{F}$, in which $h(s) = \alpha f(s) + (1 - \alpha)g(s) \in X$, for each s in S.

The decision-maker's preferences on $\mathcal{F}$ are given by a binary relation $\succsim$. Let $\succ$ denote the strict preference and $\sim$ denote indifference derived from $\succsim$ in the usual way.

Definition 1. For all acts f in $\mathcal{F}$, we say that a constant act x_f in X is a certainty equivalent of f if $x_f \sim f$.

For most models dealing with a mixture space of acts, the first step is to show that the axioms induce an expected utility representation over the set of constant acts. That is, there exists an affine function $U: X \to \mathbb{R}$, that represents $\succsim$ restricted to X. Affinity of U means that $U(\alpha x + (1 - \alpha)y) = \alpha U(x) + (1 - \alpha)U(y)$, for all x and y in X, and all α in $[0, 1]$.

Once we have obtained a utility representation of the preferences on the constant acts, it is convenient to map each act to its corresponding state-utility vector, and to consider the preference relation over these state-utility vectors induced by the underlying preferences over acts.

Thus, given an affine function $U: X \to \mathbb{R}$ on the constant acts, where 0 is in the interior of $U(X)$, we can map each act f to the state-utility vector $U \circ f \in U(X)^n$ given by $(U \circ f)_s = U(f(s))$. Recalling from the previous section that e denotes the constant vector $(1, \dots, 1) \in \mathbb{R}^n$, we will refer to the set $\{ke\colon k \in U(X)\}$ as the constant-utility line. For any given U, constant acts are mapped to state-utility vectors that lie on the constant-utility line.

The preferences $\succsim$ induce preferences on the state-utility vectors in $U(X)^n$ in the natural way.

Definition 2 *(Induced preferences).* Let $\succsim_u$ be the binary relation on $U(X)^n$ defined by: $u \succsim_u u'$ if there exists a corresponding pair of acts f and f' in $\mathcal{F}$ with $U \circ f = u$ and $U \circ f' = u'$, such that $f \succsim f'$.

Let $\Delta(S)$ denote the set of probability measures over S. For each $\pi \in \Delta(S)$, let $\pi_s := \pi(\{s\})$ for each $s \in S$ and let $\mathrm{E}_\pi(u) := \sum_s \pi_s u_s$, for each $u \in \mathbb{R}^n$. We will often refer to $\mathrm{E}_\pi(u)$ as a mean utility (of u with respect to π).

We can now formally define the family of invariant symmetric preferences.

Definition 3. An *invariant symmetric representation* is a tuple $\langle U, \pi, \rho, \varphi \rangle$ where:

(1) $U : X \to \mathbb{R}$ is an affine utility function with 0 in the interior of the range;
(2) $\pi \in \Delta(S)$ is a baseline probability;
(3) $\rho : U(X)^n \to \mathbb{R}$ is a continuous, dispersion measure with
  (a) $\rho(\lambda u) = \lambda\rho(u) \geqslant 0$, for all u in $U(X)^n$ and all $\lambda \geqslant 0$, such that λu is also in $U(X)^n$ (positive linear homogeneity),
  (b) $\rho(u + u') \leqslant \rho(u) + \rho(u')$, for all u and u' in $U(X)^n$ such that $(u + u')$ is also in $U(X)^n$ (sub-additivity),
  (c) $\rho(u + \delta e) = \rho(u)$, for all u in $U(X)^n$ and all δ in $\mathbb{R}$ such that $u + \delta e$ is also in $U(X)^n$ (translation invariance), and
  (d) $\rho(u) = \rho(-u)$, for all u in $U(X)^n$ such that $-u$ is also in $U(X)^n$ (symmetry); and
(4) $\varphi : \mathcal{D} \to \mathbb{R}$ is a continuous function, with domain $\mathcal{D} \subset U(X) \times \rho(U(X)^n)$ comprising pairs (μ', ρ') for which $\mu' = \mathrm{E}_\pi(u)$ for some $u \in \rho^{-1}(\rho')$, increasing in its first argument, non-increasing in its second argument, with $\varphi(y, 0) = y$ for all y in $U(X)$, and monotonic in u, that is,
$$\varphi\big(\mathrm{E}_\pi(u), \rho(u)\big) - \varphi\big(\mathrm{E}_\pi\big(u'\big), \rho\big(u'\big)\big) \geqslant 0,$$
for all $u \geqslant u'$ in $U(X)^n$.

The associated *invariant symmetric preferences* over acts are generated by
$$V(f) = \varphi\big(\mathrm{E}_\pi(U \circ f), \rho(U \circ f)\big), \tag{2}$$
where $U \circ f$ is the utility vector given by $(U \circ f)_s := U(f(s))$.

An invariant symmetric representation $\langle U, \pi, \rho, \varphi \rangle$ is labeled *compact* if $U(X)$ is compact (that is, closed and bounded).

In this representation, $\mathrm{E}_\pi(U \circ f)$ represents the 'mean utility' of the act f using the utility function $U(\cdot)$ and the weights π. We can think of $\rho(U \circ f)$ as a measure of dispersion of the state-utility vector $U \circ f$. The overall representation is weakly separable in the mean and dispersion, and is strictly increasing in the former and non-increasing in the latter. The normalization $\varphi(y, 0) = y$ ensures that the value of a constant act x is equal to the utility of that act, $V(x) = U(x)$, and hence that the value of a general act f is equal to the utility of its certainty equivalent, $V(f) = U(x_f)$.

Notice that both the standard deviation and the inter-quartile range satisfy (3)(a)–(d).

3.1. Uniqueness

For preferences that admit the invariant symmetric representation $\langle U, \pi, \rho, \varphi \rangle$, the baseline probability π is unique but there is some indeterminacy in specifying the other three components of the representation. Not surprisingly, the utility function U is unique only up to a positive affine transformation, while the measure of dispersion is unique up to multiplication by a positive scalar. That is, we can take a positive affine transformation of the utility function and a positive multiple of the suitably adjusted measure of dispersion function and then, with appropriate adjustments to φ, obtain another invariant symmetric representation for the same preferences.

We state the precise class of invariant symmetric representations that generate the same preferences in the following lemma.

Lemma 1 *(Uniqueness). Let* $\succsim$ *and* $\tilde{\succsim}$ *be two preference relations generated by the invariant symmetric representations* $\langle U, \pi, \rho, \varphi\rangle$ *and* $\langle \tilde{U}, \tilde{\pi}, \tilde{\rho}, \tilde{\varphi}\rangle$*, respectively. The following are equivalent*:

(1) *The two relations* $\succsim$ *and* $\tilde{\succsim}$ *are equal.*
(2) *There exist* $\alpha, \gamma > 0$ *and* $\beta \in \mathbb{R}$*, such that* $\tilde{U}(x) := \alpha U(x) + \beta$*,* $\tilde{\rho}(\tilde{u}) := \gamma\rho([\tilde{u} - \beta e]/\alpha)$ *and* $\tilde{\varphi}(\tilde{\mu}, \tilde{\rho}) := \alpha\varphi([\tilde{\mu} - \beta]/\alpha, \tilde{\rho}/\gamma) + \beta$.

To see that (2) implies (1) notice that for the representation of $\tilde{\succsim}$ corresponding to $\langle \tilde{U}, \pi, \tilde{\rho}, \tilde{\varphi}\rangle$ we have:

$$\begin{aligned}
\tilde{V}(f) &= \tilde{\varphi}\big(\mathrm{E}_\pi(\tilde{U} \circ f), \tilde{\rho}(\tilde{U} \circ f)\big) \\
&= \alpha\varphi\big(\big[\mathrm{E}_\pi(\alpha U \circ f + \beta e) - \beta\big]/\alpha, \gamma\rho\big([\alpha U \circ f + \beta e - \beta e]/\alpha\big)/\gamma\big) + \beta \\
&= \alpha\varphi\big(\mathrm{E}_\pi(U \circ f), \rho(U \circ f)\big) + \beta = \alpha V(f) + \beta.
\end{aligned}$$

The proof that (1) implies (2) is more involved but follows from standard arguments in establishing uniqueness of cardinal representations and so is omitted.

As we shall see below in Sections 6 and 7, the key properties of the representation required for the preferences to exhibit particular comparative statics behavior or to generate asset pricing formulas will be the same irrespective of which invariant symmetric representation is used.

4. Axioms and main theorem

The first two axioms below, an ordering and a continuity axiom, are standard.

A.1 *Order*. $\succsim$ is transitive and complete.
A.2 *Continuity*. For any three acts f, g and h in $\mathcal{F}$ such that $f \succsim h \succsim g$, the sets $\{\alpha \in [0, 1]: \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0, 1]: h \succsim \alpha f + (1-\alpha)g\}$ are closed.

It is also usual to have some form of monotonicity axiom that also delivers state independence and to have a non-degeneracy axiom.

A.3 *Monotonicity*. For any pair of acts f and g in $\mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.
A.4 *Best and worst outcome*. There exist outcomes z and w in X satisfying $z \succ w$ and $z \succsim x \succsim w$, for all x in $\mathcal{X}$.

The stronger axiom A.4, requiring the existence of a best and a worst outcome, is not essential in what follows but it simplifies the analysis by ensuring that the representation obtained is bounded above and below.

The next axiom builds on Siniscalchi's [23] notion of 'complementary acts'.

Definition 4. Two acts f and $\bar{f}$ are *complementary* if $\frac{1}{2}f + \frac{1}{2}\bar{f} = x$ for some x in X.

If two acts f and $\bar{f}$ are complementary then $(f, \bar{f})$ is referred to as a *complementary pair*.

Notice that two acts are complementary if a fifty–fifty mixture of the pair provides a 'perfect' hedge against subjective uncertainty.[8] Furthermore, if preferences over constant acts admit the expected utility representation U, then the state-utility vectors associated with the complementary pair of acts $(f, \bar{f})$, denoted by $U \circ f$ and $U \circ \bar{f}$ satisfy $U \circ f = 2ke - U \circ \bar{f}$ (or equivalently, $\frac{1}{2}U \circ f + \frac{1}{2}U \circ \bar{f} = ke$) for some constant $k \in U(X)$. "Thus, complementarity is the preference counterpart of algebraic negation" (Siniscalchi [23, p. 810]). And, if $(f, \bar{f})$ and $(g, \bar{g})$ are complementary pairs of acts, with $\frac{1}{2}f + \frac{1}{2}\bar{f} = x$ and $\frac{1}{2}g + \frac{1}{2}\bar{g} = y$, then, for any weight α in [0, 1], the pair $(\alpha f + (1-\alpha)g, \alpha\bar{f} + (1-\alpha)\bar{g})$ is also a complementary pair, since

$$\frac{1}{2}\big(\alpha f + (1-\alpha)g\big) + \frac{1}{2}\big(\alpha\bar{f} + (1-\alpha)\bar{g}\big)$$
$$= \alpha\left[\frac{1}{2}f + \frac{1}{2}\bar{f}\right] + (1-\alpha)\left[\frac{1}{2}g + \frac{1}{2}\bar{g}\right] = \alpha x + (1-\alpha)y.$$

As the state-utility vectors associated with a pair of complementary acts are reflections of each other in the constant-utility line 'mirror', all symmetric measures of dispersion attribute to them the same utility variability. So, if we are attributing the same utility variability to any pair of complementary acts, then we might plausibly rank such pairs of acts according to the same baseline measure. Hence, if a given pair of complementary acts $(f, \bar{f})$ are indifferent to each other, then those two acts should have the *same* mean utility according to the baseline measure. The mean utility in turn is the utility of any constant act that is indifferent to the perfect hedge $\frac{1}{2}f + \frac{1}{2}\bar{f}$. This is illustrated in Fig. 1 which plots the state-utility vectors of two complementary acts f and $\bar{f}$ that are also indifferent to each other.

The next axiom, complementary independence (due to Siniscalchi [23]), formalizes this intuition that the ranking of pairs of complementary acts should be in accordance with their expectations taken with respect to the underlying baseline measure and hence should conform to expected utility theory.

A.5 *Complementary independence*. For any two complementary pairs of acts $(f, \bar{f})$, $(g, \bar{g})$: if $f \succsim \bar{f}$ and $g \succsim \bar{g}$ then $\alpha f + (1-\alpha)g \succsim \alpha\bar{f} + (1-\alpha)\bar{g}$ for all α in (0, 1).

If we deem the constant act x to be the mean of the act f because there exists another act $\bar{f}$ that is both complementary and indifferent to f and $x = \frac{1}{2}f + \frac{1}{2}\bar{f}$, then it seems natural to consider x to be the mean of any other act h in which $h = \lambda f + (1-\lambda)x$ for some λ in (0, 1]. As we see in Fig. 1, the plot of the state-utility vector associated with such an act h lies on the ray from $U(x)e$ through $U \circ f$ and hence resides in the hyperplane through $U(x)e$ with normal

[8] Siniscalchi's definition differs from ours. He defines as complementary any pair of acts for which a fifty–fifty mixture constitutes a subjectively perfect hedge. Formally, for him any pair of acts f and $\bar{f}$ are *complementary* if, for any two states s and s',

$$\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s').$$

The advantage of our definition is that it is 'preference free'. Any pair of acts which are complementary under our definition are complementary for any decision maker. But with A.3 (monotonicity) and A.4 (existence of best and worst outcome), the set of acts that map to lotteries whose support is a subset of the best and worst outcomes is rich enough to provide enough pairs of complementary acts that enable us to derive the same implications as Siniscalchi achieved with his preference-based definition.

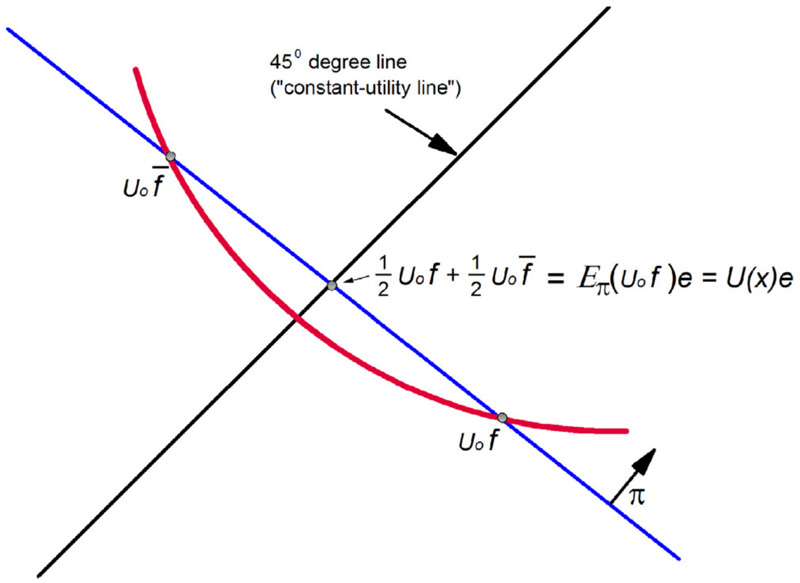

Fig. 1. An illustration of the two state-utility vectors associated with two complementary acts f and $\bar{f}$ that are indifferent to each other.

vector π. This motivates the following notion of the mean (constant) act for an act that is defined directly in terms of the underlying preferences over acts.

Definition 5 *(Mean and common mean).* A constant act x is deemed to be the *mean* for an act f if there exist an act f' and $\lambda \in (0, 1]$, such that

(i) $\lambda f + (1-\lambda)x \sim f'$; and
(ii) $\frac{1}{2}[\lambda f + (1-\lambda)x] + \frac{1}{2} f' = x$.

(Hence $(\lambda f + (1-\lambda)x, f')$ is a complementary pair of acts.)

If the constant act x is the mean of both f and g, then f and g are said to have a common mean.[9]

With this definition of the mean of an act in hand, we can now introduce the last two axioms which can readily be seen to be weakenings of Gilboa and Schmeidler's [11] uncertainty aversion axiom and certainty independence axiom, respectively. Formally, they restrict the application of those two axioms to pairs of acts that have a common mean.

A.6 *Common-mean uncertainty aversion*. For any pair of acts f and g, and any α in $(0, 1)$, if f and g have a common mean and $f \sim g$ then $\alpha f + (1-\alpha)g \succsim f$.

A.7 *Common-mean certainty independence*. For any pair of acts f, g in $\mathcal{F}$, any constant act x in X and any α in $(0, 1)$: if f and g have a common mean then

$$f \succsim g \quad \Leftrightarrow \quad \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.$$

[9] If the range $U(X)$ was unbounded, it would be enough to work with a simpler definition in which $\lambda = 1$. However, when dealing with a bounded range we can no longer ensure that for every utility vector u in $U(X)^n$ there exist a complementary vector $\bar{u}$ also in $U(X)^n$, and a complementary pair of acts $(f, \bar{f})$, satisfying $u = U \circ f$, $\bar{u} = U \circ \bar{f}$ and $f \sim \bar{f}$, thereby ensuring the existence of mean utility for every utility vector in $U(X)^n$.

We can now state our main result.

Theorem 2 *(Main theorem). The preferences* $\succsim$ *on* $\mathcal{F}$ *satisfy* A.1 (*weak order*), A.2 (*continuity*), A.3 (*monotonicity*), A4 (*best and worst outcome*), A.5 (*complementary independence*), A.6 (*common-mean uncertainty aversion*) *and* A.7 (*common-mean certainty independence*) *if and only if they admit a* compact invariant symmetric *representation* $\langle U, \pi, \varphi, \rho \rangle$.

The proof of the theorem appears in Appendix A. Notice however, that since standard uncertainty aversion is not necessary, the φ in the representation need not be concave, thus the representation allows for possibly non-convex upper contour sets and so may appear ambiguity seeking for certain mixtures of acts that come from different equal-mean sets.[10] The axioms do entail that ρ is sub-additive, which reflects the fact that the upper-contour sets of preferences restricted to a set of common-mean acts must be convex. In the next section, we provide a geometric intuition as to how common-mean certainty independence leads to a representation that is weakly separable in the mean and dispersion of the associated state-utility vector.

5. Interpretation and geometry

Given an affine utility function U on outcomes (and on constant acts) and probability weights π on the states, we may associate with the act f the state-utility vector $U \circ f$ whose mean with respect to π is $\mu := \mathrm{E}_\pi(U \circ f)$. Furthermore, we can think of the absolute uncertainty premium (measured in utility) of the act f as being the difference $\mu - U(x_f)$ between the mean utility and the utility of the certainty equivalent. For the mean-dispersion preferences that were analyzed in Grant and Polak [12], this premium was equal to the measure of dispersion $\rho(U \circ f)$. With invariant symmetric preferences, the premium is given by $\varphi(\mu, 0) - \varphi(\mu, \rho(U \circ f))$. Thus, the premium depends not only on the measure of dispersion $\rho(U \circ f)$ but also on φ which in turn depends on the mean utility μ.

Figures 2–4 illustrate how the key axiom, common-mean certainty independence, allows uncertainty premiums to vary. They are drawn for the case where the induced preferences over state-utility vectors are smooth. Suppose f and g are a pair of acts, with common mean x, which are indifferent to each other. The associated state-utility vectors $U \circ f$ and $U \circ g$ are plotted in Fig. 2. Since f and g have common mean x, the associated state utility vectors $U \circ f$ and $U \circ g$ must both lie in the hyperplane through μe, where $\mu = U(x)$. Let π denote its normal vector. The certainty equivalent constant utility vector corresponds to the point $V(f)e$ where the indifference set of $\succsim_u$ in which $U \circ f$ and $U \circ g$ both reside intersects the constant-utility line. The uncertainty premium (measured in utils) is given by $\mu - \varphi(\mu, \rho(U \circ f))$.

Now for fixed α in $(0, 1)$, consider the two acts $\alpha f + (1 - \alpha)x$ and $\alpha g + (1 - \alpha)x$ formed by taking convex combinations of the common mean x with f and with g, respectively. Applying axiom A.7, it follows that the state-utility vectors $\alpha U \circ f + (1 - \alpha)\mu e$ and $\alpha U \circ g + (1 - \alpha)\mu e$ which are plotted in Fig. 3 must lie on the same indifference curve. As α was arbitrary, and the same applies for any pair of acts that have x as a common mean, this in turn means that the indifference map on the hyperplane through μe with normal vector π is *homothetic*.

[10] In Section 6 we show how such departures from convexity allow us to accommodate a preference pattern in an example of Machina [16] that Baillon et al. [2] show is problematic for any preference relation that satisfies Gilboa and Schmeidler's [11] uncertainty aversion.

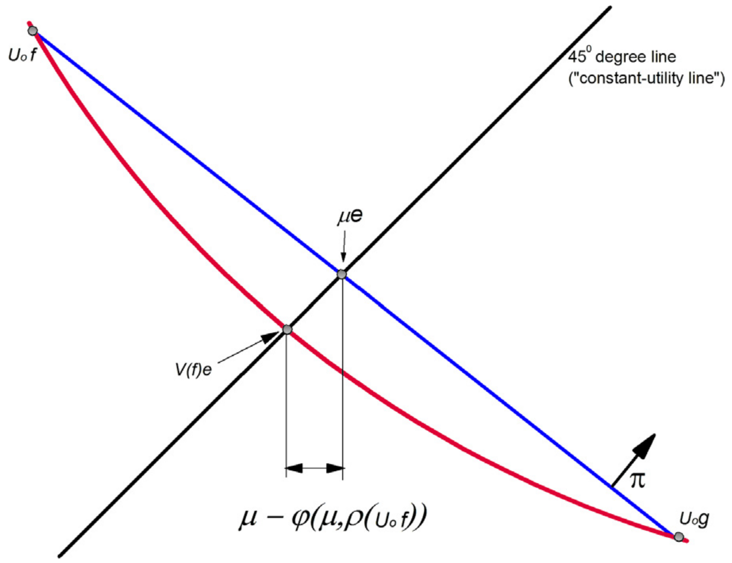

Fig. 2. Illustration of uncertainty premium.

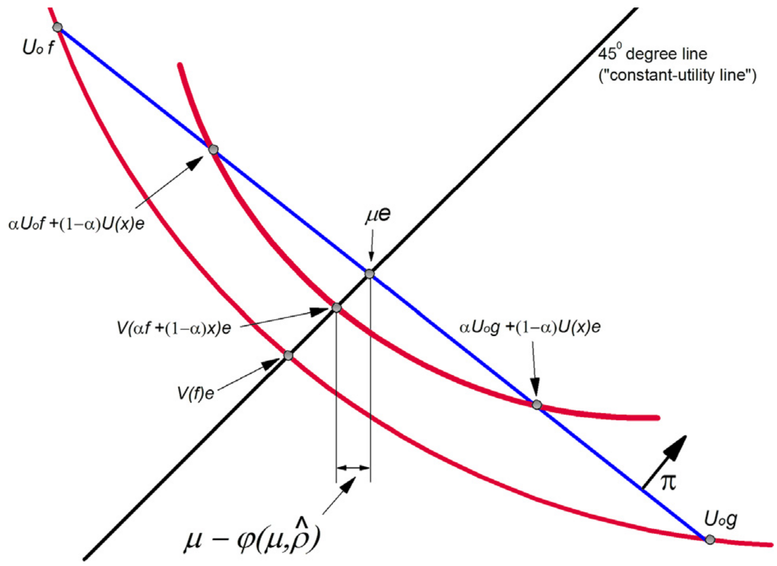

Fig. 3. Illustration that indifference curves on an equal-mean hyperplane are homothetic.

More generally Lemma 13 in Appendix A implies the following scale invariance property of the induced preferences.

Definition 6 *(Common-mean radial homotheticity).* There exists a probability $\pi \in \Delta(S)$, such that for any pair of utility vectors u' and u'' in $U(X)^n$, for which $\mathrm{E}_\pi(u') = \mathrm{E}_\pi(u'')$: $u' \succsim_u u''$ if and only if $\alpha u' + (1-\alpha)\mathrm{E}_\pi(u')e \succsim_u \alpha u'' + (1-\alpha)\mathrm{E}_\pi(u'')e$, for all $\alpha \in (0, 1)$.

Next consider some other constant act y and the two new acts $\alpha f + (1-\alpha)y$ and $\alpha g + (1-\alpha)y$, formed by taking convex combinations of y with f and with g, respectively. Once again, since f and g have a common mean, we can apply axiom A.7. Hence the state-utility vectors $\alpha U \circ f + (1-\alpha)U(y)e$ and $\alpha U \circ g + (1-\alpha)U(y)e$ which are plotted in Fig. 4 must also lie on the same indifference curve. Again, by construction, the two new state-utility vectors $\alpha U \circ f + (1-\alpha)U(y)e$ and $\alpha U \circ g + (1-\alpha)U(y)e$ have the same mean with respect to π;

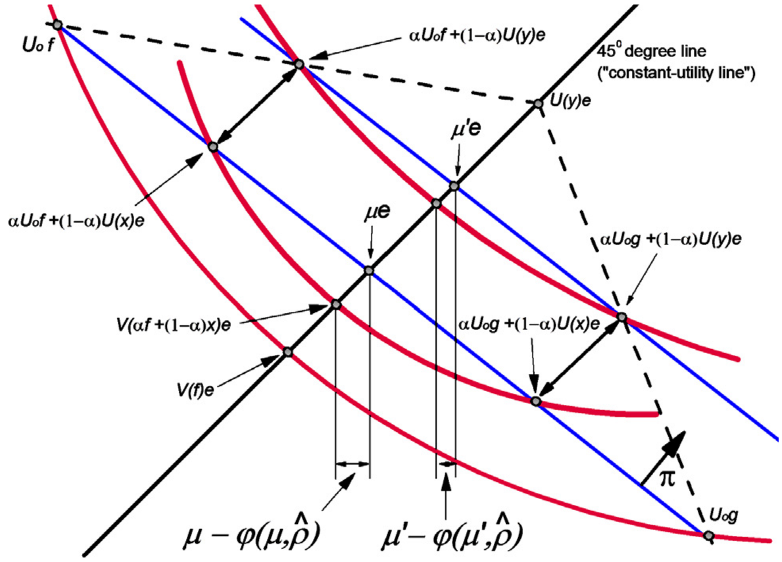

Fig. 4. Illustration of invariant complementary-symmetric preferences with non-constant absolute uncertainty premium.

in this case μ'. In fact, each of these two new indifferent vectors is obtained from the previous pair of indifferent state-contingent utility vectors by the common translation $(1-\alpha)(U(y) - U(x))e$ (that is, a translation parallel to the constant-utility line). More generally, Lemma 14 in Appendix A shows that axiom A.7 implies the following translation invariance property of the induced preferences $\succsim_u$.

Definition 7 *(Common-mean translation invariance).* There exists a probability $\pi \in \Delta(S)$, such that for any pair of utility vectors u' and u'' in $U(X)^n$, for which $\mathrm{E}_\pi(u') = \mathrm{E}_\pi(u'')$ and any $\delta \in \mathbb{R}$, for which $u' + \delta e$ and $u'' + \delta e$ are both in $U(X)^n$: $u' \succsim_u u''$ if and only if $u' + \delta e \succsim_u u'' + \delta e$.

Although axiom A.7 is not weaker than Maccheroni et al.'s [15] weak certainty independence axiom, the property of common-mean translation invariance is weaker than the translation invariance property implied by weak certainty independence. In particular there is no requirement that, if we apply the same common translation $(1-\alpha)(U(y) - U(x))e$ to the entire indifference curve through $V(\alpha f + (1-\alpha)x)e$, then all points in the new translated curve will be indifferent. The reason is that not all the points on the original indifference curve had the same mean. In our picture, the actual indifference curve through $\alpha U \circ f + (1-\alpha)U(y)e$ is less bowed.

Now consider uncertainty premiums. The mean of the two vectors $\alpha f + (1-\alpha)x$ and $\alpha g + (1-\alpha)x$ was $\mu = \varphi(\mu, 0)$. Since they had the same mean and were indifferent, they must have the same dispersion term $\hat{\rho}$: that is, the utility of their certainty equivalent is $V(\alpha f + (1-\alpha)x) = V(\alpha g + (1-\alpha)x) = \varphi(\mu, \hat{\rho})$. Thus the uncertainty premium associated with those two vectors is just $\mu - \varphi(\mu, \hat{\rho})$. The mean of the two vectors $\alpha f + (1-\alpha)y$ and $\alpha g + (1-\alpha)y$ was μ'. By construction, they had the same vector of differences from this mean as the other two vectors, hence their dispersion term was also $\hat{\rho}$. Thus, the utility of their certainty equivalent $V(\alpha f + (1-\alpha)y) = V(\alpha g + (1-\alpha)y) = \varphi(\mu', \hat{\rho})$, and their uncertainty premium is just $\mu' - \varphi(\mu', \hat{\rho})$. But, as shown, these premiums need not be the same: in the illustrated case, the uncertainty premium decreased as we increased the mean utility holding the dispersions fixed.

6. Absolute uncertainty aversion

In the analysis of risk, one way to define decreasing absolute risk aversion is (abusing our notation): for all random variables $\tilde{X}$, $\tilde{Y}$ such that $\tilde{X}$ is riskier than $\tilde{Y}$ in some sense, if $\tilde{X}$ is weakly preferred to $\tilde{Y}$ then for any $\delta > 0$, the 'improved' random variable $\tilde{X} + \delta$ is also weakly preferred to the improved random variable $\tilde{Y} + \delta$.[11] That is, the set of acceptable increases in risk can only expand and not contract as non-state contingent wealth is increased. Increasing absolute risk aversion can be defined similarly.

For the family of preferences defined over subjectively uncertain acts considered here, we can define analogous concepts of decreasing absolute uncertainty aversion and increasing absolute uncertainty aversion.

We begin by proposing one notion of what it might mean for one act to be deemed 'more dispersed' than another. In particular, we shall propose that, if one act can be expressed as a convex combination of another act and a constant act, then the latter act is deemed more dispersed than the former, since the former act is 'between' (in terms of mixtures) the constant act (that by definition has zero dispersion) and the latter act.

Definition 8 *("At least as dispersed as" partial ordering).* An act f is considered *at least as dispersed as* the act g, denoted $f \trianglerighteq g$, if there exist a constant act x and a $\lambda \in [0, 1]$, such that $g = \lambda f + (1 - \lambda)x$.

The relation $\trianglerighteq$ respects Gilboa and Schmeidler's [11] Certainty Independence axiom.

Proposition 3. *For any pair of acts f and g in $\mathcal{F}$, any constant act y in X and any α in $(0, 1)$: $f \trianglerighteq g$ if and only if $\alpha f + (1 - \alpha)y \trianglerighteq \alpha g + (1 - \alpha)y$.*

With the *at least as dispersed* (*partial*) *ordering* $\trianglerighteq$ in hand, we can now define the corresponding notions of decreasing, increasing and constant absolute uncertainty aversion.

Definition 9 *(DAUA, IAUA and CAUA).* We say that $\succsim$ exhibits decreasing absolute uncertainty aversion (DAUA) if, for any pair of acts f and g in $\mathcal{F}$, such that $f \trianglerighteq g$, any pair of constant acts x and y, such that $y \succsim x$, and any α in $(0, 1)$:

$$\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x \quad \Rightarrow \quad \alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y. \tag{3}$$

We say the agent exhibits increasing absolute uncertainty aversion (IAUA) if expression (3) holds for any constant acts x and y such that $x \succsim y$. And we say the agent exhibits constant absolute uncertainty aversion (CAUA) if she exhibits both DAUA and IAUA.

The following proposition characterizes the class of invariant symmetric preferences that exhibit DAUA (respectively, IAUA).

Proposition 4 *(DAUA). Suppose that the preferences $\succsim$ admit the invariant symmetric representation $\langle U, \pi, \rho, \varphi \rangle$. Then the following two properties are equivalent*:

---

[11] Properly speaking this should be referred to as *non-increasing* absolute risk aversion, but we will follow the common usage in the risk literature.

(a) *The preferences* $\succsim$ *exhibit DAUA* (*respectively, IAUA*).
(b) *Whenever* $\varphi(\mu,\rho)=\varphi(\mu',\rho')$ *and* $\mu>\mu'$ *then* $\varphi(\mu+\delta,\rho)\geqslant$ (*resp.* $\leqslant$) $\varphi(\mu'+\delta,\rho')$ *for all* $\delta>0$, *such that for some* $\bar{u}'$ *in* $\rho^{-1}(\rho')$, $\mathrm{E}_\pi(\bar{u}')=\mu'+\delta$.

Furthermore, assuming φ *is twice differentiable,* (b) *is equivalent to*

$$\frac{-\varphi_{11}}{\varphi_1}\leqslant (\textit{resp.}\ \geqslant)\ \frac{\varphi_{12}}{-\varphi_2}. \tag{4}$$

The left-hand side of inequality (4) resembles the Arrow–Pratt coefficient of absolute risk aversion from expected utility theory and measures the concavity of φ with respect to its first argument, μ. It is also the negative of the semi-elasticity of φ_1 with respect to μ. Similarly, the right-hand side of inequality (4) is the semi-elasticity of φ_2 with respect to μ. Analogous to Pratt's analysis of risk aversion in the small, we have that the invariant symmetric preferences exhibit DAUA (respectively, IAUA) locally if the negative of the semi-elasticity of φ_1 with respect to μ is less than or equal to (respectively, is greater than or equal to) the semi-elasticity of φ_2 with respect to μ.

For (additively) separable φ, that is, where $\varphi_{12}=0$, applying inequality (4) yields that DAUA holds if and only if φ is convex in μ, and IAUA holds if and only if φ is concave in μ. By combining these last two implications we have:

Corollary 5. *Suppose that the preferences* $\succsim$ *admit the invariant symmetric representation* $\langle U,\pi,\rho,\varphi\rangle$. *Then the following are equivalent*: (i) *preferences exhibit CAUA*; *and* (ii) $\varphi(\mu,\rho)=\mu-\phi(\rho)$, *for some increasing function* $\phi(\cdot)$.

Notice that for this subclass of invariant symmetric preferences that exhibit CAUA, although $\rho(u)$ is a convex function of u, in general $\phi\circ\rho(u)$ need not be convex. But if ϕ is not convex then it follows that $\mathrm{E}_\pi(u)-\phi\circ\rho(u)$ is not concave, and thus we see there are invariant symmetric preferences that do not satisfy Gilboa and Schmeidler's [11] property of uncertainty aversion.

In many applications it may be convenient to require an invariant symmetric preference relation to exhibit (full) uncertainty aversion but, as Baillon et al. [2] have shown, this can be problematic if we also want it to accommodate the examples proposed by Machina [16]. For instance, in Machina's 50:51 example there is a urn with 101 balls, fifty of which are marked with a '1' or with a '2', with the remaining fifty-one marked either with a '3' or with a '4'. Let s_n denote the 'state' in which a ball marked with an $n\in\{1,2,3,4\}$ is drawn. Following Baillon et al. [2] we display in Table 1 the outcomes of the four acts from Machina's example expressed in 'utils'. In addition, the final two columns specify the mean utility using the natural baseline probability weights

$$(\pi_1,\pi_2,\pi_3,\pi_4)=(25/101,25/101,51/202,51/202),$$

and a measure of dispersion $\rho(U\circ f)$, specified by

$$\rho(u)=(\pi_1+\pi_2)|u_1-u_2|+(\pi_3+\pi_4)|u_3-u_4|.$$

Notice that the state-utility vector associated with act f_2 (respectively, f_4) can be obtained from the state-utility vector associated with act f_1 (respectively, f_3) by swapping the outcomes on states s_2 and s_3. If the individual is sufficiently averse to 'ambiguous' odds then Machina suggests she will prefer f_1 to f_2 since f_1 is 'unambiguous' (reflected by $\rho(U\circ f_1)=0$) and f_2

Table 1
The 50:51 example.

Acts	50 balls		51 balls		$\mathrm{E}_\pi(U \circ f)$	$\rho(U \circ f)$
	s_1	s_2	s_3	s_4		
$U \circ f_1$	202	202	101	101	151	0
$U \circ f_2$	202	101	202	101	$151 \cdot 5$	101
$U \circ f_3$	303	202	101	0	$150 \cdot 5$	101
$U \circ f_4$	303	101	202	0	151	202

is 'ambiguous' (reflected by $\rho(U \circ f_2) = 101$) with only a slightly higher mean utility than f_1 to compensate. For the pair of acts f_3 and f_4, f_4 has higher dispersion than f_3 as well as a slightly higher mean utility. However, it does not seem self-evident that the individual should necessarily prefer f_3 to f_4. To accommodate the preference pattern $f_1 \succ f_2$ and $f_4 \succ f_3$ requires:

$$\mathrm{E}_\pi(U \circ f_1) - \phi(U \circ f_1) > \mathrm{E}_\pi(U \circ f_2) - \phi(U \circ f_2),$$
$$\mathrm{E}_\pi(U \circ f_4) - \phi(U \circ f_4) > \mathrm{E}_\pi(U \circ f_3) - \phi(U \circ f_3)$$

which simplifies to:

$$\phi(101) - \phi(0) > \frac{1}{2} > \phi(202) - \phi(101).$$

This pair of inequalities clearly cannot hold for a ϕ that is convex, but if we only require ϕ to be increasing, which is all that is implied by common-mean uncertainty aversion, then the Machina 50:51 example can readily be accommodated.[12] Our example is also a member of the vector expected utility (VEU) family of preferences, axiomatized by Siniscalchi [23]. He provides an example of a VEU preference relation that can rationalize Machina's [16] reflection example. It would be straightforward to adapt our example to rationalize the reflection example as well.

Turning to preferences that exhibit a non-constant degree of absolute uncertainty aversion, we can derive the implication of decreasing absolute uncertainty premiums as defined by $\varphi(\mu, 0) - \varphi(\mu, \rho)$.

Corollary 6. *Suppose that the preferences $\succsim$ admit the invariant symmetric representation $\langle U, \pi, \rho, \varphi \rangle$ and exhibit DAUA (respectively, IAUA). Then, for all $\bar{\rho}$ in $\rho(U(X)^n)$,*

(a) *the absolute uncertainty premium $[\varphi(\mu, 0) - \varphi(\mu, \bar{\rho})]$ is non-increasing (respectively, non-decreasing) in μ; and*
(b) $\varphi(\mu + \delta, \bar{\rho}) \geqslant$ *(resp. $\leqslant$)* $\varphi(\mu, \bar{\rho}) + \delta$.

One set of examples of invariant symmetric preferences that allow for varying premiums are those that correspond to Epstein's [5] generalized mean-variance preferences (translated from risk to uncertainty). But those preferences also violate monotonicity. The following is an example of invariant symmetric preferences that exhibit decreasing absolute uncertainty aversion, but are monotone.

[12] We thank the editor for suggesting that the gap between uncertainty aversion and common-mean uncertainty aversion allows for the existence of members of the family of invariant symmetric preferences that can accommodate the Machina examples.

Example 1. Consider the mean-dispersion representation $\langle U, \pi, \rho, \varphi\rangle$ where U is a bounded affine utility function in which $U(w) = -1$ and $U(z) = 1$; π is a probability; $\rho(u) := \sum_s \pi_s |u_s - \mathrm{E}_\pi(u)|$; and $\varphi(\mu, \rho) := \mu - \kappa(\mu)\log(1 + \rho)$ where $\kappa : U(X) \to [0, 1]$ is a twice-differentiable function with $\kappa' < 0$, $\kappa'' < 0$, $\kappa(-1) = 1/4$ and $\kappa(1) = 0$.

This example may be viewed as a generalization of preferences introduced by Ergin and Gul [6]. Ergin and Gul's preferences have $\kappa(\mu) \equiv 1/4$, and hence are quasi-linear in μ, and therefore exhibit constant absolute uncertainty aversion. The preferences in this example are only weakly separable since μ appears in the term $\kappa(\mu)$. It is straightforward to see that these preferences exhibit the property that the absolute uncertainty premium is decreasing in μ since the weight $\kappa(\mu)$ put on $\log(1 + \rho)$ is decreasing in μ.[13] They also exhibit DAUA since

$$\frac{-\varphi_{11}(\mu,\rho)}{\varphi_1(\mu,\rho)} = \frac{\kappa''(\mu)\log(1+\rho)}{1-\kappa'(\mu)\log(1+\rho)} \leqslant 0 < \frac{-\kappa'(\mu)}{\kappa(\mu)} = \frac{\varphi_{12}(\mu,\rho)}{-\varphi_2(\mu,\rho)},$$

ensures that inequality (4) holds everywhere.

7. Two-fund separation and an asset pricing formula

Choice problems for individuals with invariant symmetric preferences are especially convenient analytically, most notably in the canonical portfolio problem, where a version of two-fund separation applies and which admits a CAPM style pricing formula. Two features of the model are of particular relevance. First, in many cases of interest, the baseline probability π derived from observational data may best be represented by a fat-tailed probability distribution with infinite variance. The model presented here is sufficiently flexible to incorporate dispersion measures appropriate to problems of this kind.[14] Second, as is illustrated by the problem of home country bias, investors may display ambiguity aversion, so that investments in the home market may be regarded as having lower dispersion ρ than foreign investments with the same return distribution according to the baseline probability π. It follows that investors may prefer either side of an investment in the home market to either side of an investment in the foreign market.

For ease of exposition, we shall assume that X is the real line, and that the decision-maker's preferences admit an invariant symmetric representation $\langle U, \pi, \rho, \varphi\rangle$.[15] Because our interest is in the investor's preferences over trade-offs between expected return and the state-contingent dispersion of returns, we take the affine utility U to be the identity function $I(x) \equiv x$. In this context, the assumption that attitudes to ambiguity may vary with wealth, and in particular that preferences should display decreasing absolute uncertainty aversion seems particularly appealing.

Given the assumption $U(x) = I(x)$, we may identify the choice set F (the set of feasible state-contingent returns) with the set of state-contingent incomes. Furthermore, suppose F is generated by the set of portfolios made up from a set of assets, one of which is a safe asset with return vector re. More precisely, let $r^j \in \mathbb{R}^n$, $j = 0, 1, \dots, J$, be the return vector on asset j, and let $\alpha^j \in \mathbb{R}$ be the holding of asset j, with (normalized) price equal to 1. Let asset 0 be the safe

[13] Since $\kappa(\mu) \leqslant 1/4$ for all μ in $U(X)$, it follows from Ergin & Gul's result that the preferences in this example are monotonic.

[14] Indeed Wang et al. [25] derive a CAPM-style pricing formula for symmetric stable distributed assets.

[15] This is not strictly within the framework of our characterization but we consider the natural extension of our model to this setting of unbounded utility.

asset with return r, so that $r^0 = re$. For $j = 1, \ldots, J$, set $\bar{r}^j := E_\pi(r^j)$, that is, the (subjective) expected return of asset j from the perspective of the investor.

Denote holdings of the non-safe assets by $\alpha = (\alpha^1, \ldots, \alpha^J)$ and the holding of the safe asset by α^0. The portfolio problem for an investor with initial wealth W is thus:

$$\max_{\langle \alpha^0, \alpha \rangle} \left\{ \varphi\big(E_\pi(y), \rho(y)\big):\ \alpha^0 + \sum_{j=1}^J \alpha^j = W,\ y = \alpha^0 re + \sum_{j=1}^J \alpha^j r^j \right\}$$
$$= \max_{\langle \alpha \rangle} \varphi\left(r\left(W - \sum_{j=1}^J \alpha^j \right) + \sum_{j=1}^J \alpha^j \bar{r}^j, \rho\left(\sum_{j=1}^J \alpha^j r^j \right)\right),$$

after using the translation invariance of ρ.

Letting $\bar{r}$ denote the expected return (per dollar invested) of the portfolio, then because φ is increasing in μ and decreasing in ρ, this optimization problem can be rewritten as:

$$\max_\mu \big\{ \varphi\big(\mu, \hat{\rho}(\mu, F)\big) \big\},$$

where,

$$\hat{\rho}(W\bar{r}, F) = \min_{\langle \alpha \rangle} \left\{ \rho\left(\sum_{j=1}^J \alpha^j r^j \right):\ W\bar{r} = \left(W - \sum_{j=1}^J \alpha^j \right) r + \sum_{j=1}^J \alpha^j \bar{r}^j \right\}$$
$$= \min_{\langle \alpha \rangle} \left\{ \rho\left(\sum_{j=1}^J \alpha^j r^j \right):\ W(\bar{r} - r) = \sum_{j=1}^J \alpha^j \big(\bar{r}^j - r\big) \right\}$$
$$= \begin{cases} W(\bar{r} - r) \min_{\langle \alpha \rangle} \{ \rho(\sum_{j=1}^J \alpha^j r^j):\ 1 = \sum_{j=1}^J \alpha^j (\bar{r}^j - r) \} & \text{if } \bar{r} > r, \\ 0 & \text{if } \bar{r} = r \end{cases}$$
$$= \begin{cases} W(\bar{r} - r)\hat{\rho}(1, F_{-0} + \{-re\}) & \text{if } \bar{r} > r, \\ 0 & \text{if } \bar{r} = r. \end{cases} \tag{5}$$

That is,

$$\hat{\rho}\big(1, F_{-0} + \{-re\}\big) := \min_{\langle \alpha \rangle} \left\{ \rho\left(\sum_{j=1}^J \alpha^j r^j \right):\ 1 = \sum_{j=1}^J \alpha^j \big(\bar{r}^j - r\big) \right\},$$

where $F_{-0} + \{-re\}$ denotes the set of feasible state contingent 'excess' return vectors that can be achieved through the choice of a portfolio of non-safe assets (that is, a portfolio with a zero holding of the safe asset) and where the excess return of an asset is the difference between its return and that of the safe asset.

This decomposition shows that if all investors use the same baseline measure π and the same measure of dispersion $\rho(\cdot)$, then regardless of their attitudes towards mean and dispersion as encoded in the aggregator φ, any interior solution to the investment problem (that is, where an investor chooses a portfolio with an expected return $\bar{r} > r$) satisfies two-fund separation (the mutual-fund principle). Each investor h whose preferences admit the invariant symmetric representation $\langle I, \pi, \rho, \varphi^h \rangle$ chooses some linear combination of the safe asset and the portfolio of non-safe assets defined by:

$$\hat{\alpha} \in \arg\min \left\{ \rho\left(\sum_{j=1}^J \alpha^j u^j \right):\ 1 = \sum_{j=1}^J \alpha^j \big(\bar{r}^j - r\big) \right\}. \tag{6}$$

If $\rho(\cdot)$ is 'smooth enough' to have a gradient $\nabla\rho$,[16] the first-order conditions for (6) require for each j, evaluated at the optimal portfolio $\hat{\alpha}$:

$$\left\langle \nabla\rho\left(\sum_{k=1}^{J}\hat{\alpha}^k r^k\right), r^j\right\rangle - \lambda\left(\bar{r}^j - r\right) = 0, \tag{7}$$

where λ is the associated Lagrangian multiplier and $\langle v, v'\rangle$ denotes the inner product of the two vectors v and v'. Multiplying both sides of (7) by $\hat{\alpha}^j$ and summing, we obtain:

$$\lambda\sum_{j=1}^{J}\hat{\alpha}^j\left(\bar{r}^j - r\right) = \left\langle \nabla\rho\left(\sum_{k=1}^{J}\hat{\alpha}^k r^k\right), \sum_{j=1}^{J}\hat{\alpha}^j r^j\right\rangle = \rho\left(\sum_{j=1}^{J}\hat{\alpha}^j r^j\right),$$

where the second equality follows by the linear homogeneity of ρ. Recall from the constraint for (6) that $\sum_{j=1}^{J}\hat{\alpha}^j(\bar{r}^j - r) = 1$, hence substituting for λ in (7) gives for each j:

$$\bar{r}^j - r = \frac{\langle\nabla\rho(\sum_{k=1}^{J}\hat{\alpha}^k r^k), r^j\rangle}{\rho(\sum_{j=1}^{J}\hat{\alpha}^j r^j)}. \tag{8}$$

Let $\bar{r}^M$ denote the expected return per dollar spent on the efficient mutual fund. By definition,

$$\bar{r}^M = \sum_{j=1}^{J}\gamma^j\bar{r}^j, \quad \text{where } \gamma^j = \frac{\hat{\alpha}^j}{\sum_{k=1}^{J}\hat{\alpha}^k}.$$

Notice that γ^j is the fraction of each dollar spent on the efficient mutual fund that is used to purchase asset j.

To express (8) in a more familiar form, first notice that:

$$\bar{r}^M - r = \sum_{j=1}^{J}\gamma^j\left(\bar{r}^j - r\right) = \frac{\sum_{j=1}^{J}\hat{\alpha}^j(\bar{r}^j - r)}{\sum_{k=1}^{J}\hat{\alpha}^k}.$$

Also from the homogeneity of degree zero of $\nabla\rho$ and the linear homogeneity of ρ, we have:

$$\nabla\rho\left(\sum_{k=1}^{J}\hat{\alpha}^k r^k\right) = \nabla\rho\left(\frac{\sum_{k=1}^{J}\hat{\alpha}^k r^k}{\sum_{k=1}^{J}\hat{\alpha}^k}\right) = \nabla\rho\left(\sum_{k=1}^{J}\gamma^k r^k\right), \quad \text{and}$$

$$\frac{\rho(\sum_{j=1}^{J}\hat{\alpha}^j r^j)}{\sum_{k=1}^{J}\hat{\alpha}^k} = \rho\left(\frac{\sum_{j=1}^{J}\hat{\alpha}^j r^j}{\sum_{k=1}^{J}\hat{\alpha}^k}\right) = \rho\left(\sum_{j=1}^{J}\gamma^j r^j\right).$$

So, if we multiply the right-hand side of (8) by $\sum_{j=1}^{J}\hat{\alpha}^j(\bar{r}^j - r)$ $(= 1)$, and then multiply and divide it by $\sum_{k=1}^{J}\hat{\alpha}^k$, we obtain:

$$\bar{r}^j - r = \frac{\langle\nabla\rho(\sum_{k=1}^{J}\hat{\alpha}^k r^k), r^j\rangle}{\rho(\sum_{j=1}^{J}\hat{\alpha}^j r^j)/\sum_{k=1}^{J}\hat{\alpha}^k}\frac{\sum_{j=1}^{J}\hat{\alpha}^j(\bar{r}^j - r)}{\sum_{k=1}^{J}\hat{\alpha}^k} = \frac{\langle\nabla\rho(\sum_{k=1}^{J}\gamma^k r^k), r^j\rangle}{\rho(\sum_{j=1}^{J}\gamma^j r^j)}\left(\bar{r}^M - r\right).$$

[16] Since $\rho(\cdot)$ is convex, the gradient will exist almost everywhere. Where it does not exist there will exist a subdifferential and we can use a one-sided directional derivative instead.

Thus for each asset j:

$$\bar{r}^j = r + \beta^j\left(\bar{r}^M - r\right), \quad \text{where } \beta^j = \frac{\langle \nabla\rho(\sum_{k=1}^J \gamma^k r^k), r^j\rangle}{\rho(\sum_{j=1}^J \gamma^j r^j)}. \tag{9}$$

The interpretation of β^j as the 'generalized beta' parallels the standard CAPM model. Each asset's generalized beta measures the ratio of the increase in dispersion by spending an extra dollar on that asset to the increase in dispersion by spending an extra dollar on the efficient mutual fund $(\gamma^1, \ldots, \gamma^J)$.[17]

8. Concluding comments

The simple risk-return analysis of choice under uncertainty was derived under the highly restrictive assumption of mean-variance preferences with known probabilities. Expected utility theory provided a more general framework for analysis, but yielded more limited comparative static results, particularly in relation to background risk.

In this paper, we have shown that the core of this analysis may be extended to encompass more realistic models of preferences, in which individuals are not assumed to be (second-order) probabilistically sophisticated (in the sense of Ergin and Gul [6]), and in which the strong EU independence axiom is replaced with the weaker requirement of complementary independence, common-mean uncertainty aversion and common-mean certainty independence. The class of models described in this way includes the mean-variance model as a special case, but allows for a wide variety of dispersion measures.

From the viewpoint of economists interested in modelling problems involving uncertainty, rather than in the axiomatic details of decision theory, our message is a positive one. Under fairly general conditions, the standard economic logic of choice, applied to appropriate measures of mean return and more general measures of dispersion, yield analogs of the standard results.

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[17] As a quick check we see that for the case where the measure of dispersion is the standard deviation, that is, $\rho(u) = \sqrt{\sum_{s=1}^n \pi_s [u_s - E_\pi(u)]^2}$, the gradient is given by:

$$\nabla\rho(u) = \frac{(\pi_1[u_1 - E_\pi(u)], \ldots, \pi_n[u_n - E_\pi(u)])}{\sqrt{\sum_{s=1}^n \pi_s[u_s - E_\pi(u)]^2}}.$$

Hence,

$$\begin{aligned}\beta^j &= \frac{\sum_{s=1}^n \pi_s[(\sum_{k=1}^J \gamma^k u_s^k) - E_\pi(\sum_{k=1}^J \gamma^k u^k)]u_s^j}{\sum_{s=1}^n \pi_s[(\sum_{k=1}^J \gamma^k u_s^k) - E_\pi(\sum_{k=1}^J \gamma^k u^k)]^2} \\ &= \frac{\mathrm{COV}[(\sum_{k=1}^J \gamma^k u^k)u^j]}{\mathrm{VAR}[\sum_{k=1}^J \gamma^k u^k]}\end{aligned}$$

which corresponds to the β^j in the standard CAPM formula for asset returns.

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Appendix A. Proofs

Proof of the main theorem (Theorem 2). (*Sufficiency of axioms*) In the following, we assume A.1 (weak order), A.2 (continuity), A.3 (monotonicity) and A.4 (best and worst outcome) are given.

We first find an affine utility representation for the preferences restricted to constant acts.

Lemma 7 *(Expected utility on constant acts). Given* A.5 (*complementary independence*)*, the restriction of preferences to constant acts admits an expected utility representation. That is, there exists a mixture continuous affine utility function* $U : X \to \mathbb{R}$*, with* $U(w) = -1$*,* $U(z) = 1$ *and such that, for all* x, y *in* X*,* $U(x) \geqslant U(y)$ *if and only if* $x \succsim y$*.*

Proof. Since any pair of constant acts constitutes, by definition, a pair of complementary acts, consider any three constant acts x, x' and x''. Take $f = x$, $\bar{f} := x'$ and $g = \bar{g} = x''$, and apply the axiom. It follows that, if $x \succsim x'$, then $\alpha x + (1-\alpha)x'' \succsim \alpha x' + (1-\alpha)x''$, for all α in $(0, 1)$. That is, the restriction of $\succsim$ to X satisfies the standard independence axiom and along with A.1 to A.4, the hypotheses of the expected utility theorem. As the function U is unique up to affine transformations, we, can without loss of generality, set $U(w) := -1$ and $U(z) := 1$. □

Let x_0 denote a constant act for which $U(x_0) = 0$.

Recall the following definition from Section 3:

Definition 10 *(Induced preferences).* Let $\succsim_u$ be the binary relation on $[-1, 1]^n$ defined by $u \succsim_u u'$ if there exists a corresponding pair of acts f and f' in F with $U \circ f = u$ and $U \circ f' = u'$, such that $f \succsim f'$.

Lemma 8 *(State-utility preferences). Let* $U(.)$ *be an affine representation of* $\succsim$ *on* X *with* $U(X) = [-1, 1]$*. The binary relation* $\succsim_u$ *inherits order, continuity and monotonicity. In particular,* $u \succsim_u u'$ *if and only if for all acts* f *and* f' *in* $\mathcal{F}$ *such that* $U \circ f = u$ *and* $U \circ f' = u'$*, we have* $f \succsim f'$*.*

Proof. Completeness follows from the affineness of U. That is, for any u in $[-1, 1]^n$, there exists an act f in $\mathcal{F}$ with $U \circ f = u$, for example, the act f where $f(s) := z(u_s + 1)/2 + w(-u_s + 1)/2$. Furthermore, for any pair of acts f and g, if $U \circ f = U \circ g$, then $f(s) \sim g(s)$ for all s in S, hence by A.3 (monotonicity) $f \sim g$. Hence $U \circ f \succsim_u U \circ g$ if and only if $f \succsim g$. Similarly, $U \circ f \succ U \circ g$ if and only if $f \succ g$ and $U \circ f \sim_u U \circ g$ if and only if $f \sim g$. Hence transitivity and monotonicity are inherited by $\succsim_u$. To establish continuity, fix three utility vectors u, u' and u'' in $U \circ \mathcal{F}$ (that is, in $[-1, 1]^n$) such that $u' \succsim_u u \succsim u''$. We need to show that the sets $\{\alpha \in [0, 1]\colon \alpha u' + (1-\alpha)u'' \succsim_u u\}$ and $\{\alpha \in [0, 1]\colon u \succsim_u \alpha u' + (1-\alpha)u''\}$ are closed. Let f, f' and f'' be such that $U \circ f = u$, $U \circ f' = u'$, and $U \circ f'' = u''$. We have $\alpha u' + (1-\alpha)u'' \succsim_u u$ if and only if $aU \circ f' + (1-\alpha)U \circ f'' \succsim_u U \circ f$ if and only if $U \circ (af' + (1-\alpha)f'') \succsim_u U \circ f$ if and only if $af' + (1-\alpha)f'' \succsim f$. Hence the set $\{\alpha \in [0, 1]\colon \alpha u' + (1-\alpha)u'' \succsim_u u\}$ is equal to the set $\{\alpha \in [0, 1]\colon \alpha f' + (1-\alpha)f'' \succsim f\}$, and the set $\{\alpha \in [0, 1]\colon u \succsim_u \alpha u' + (1-\alpha)u''\}$ is equal to the set $\{\alpha \in [0, 1]\colon f \succsim_u \alpha f' + (1-\alpha)f''\}$, which are closed by A.2. □

We next show that every act has a certainty equivalent.

Definition 11. For all acts f in $\mathcal{F}$, we say that a constant act x_f in X is a certainty equivalent of f if $x_f \sim f$.

Lemma 9 *(Certainty equivalents). Given* A.5 (*complementary independence*), *all acts f in $\mathcal{F}$ have certainty equivalents.*

Proof. To see this, fix f and consider $u^f = U \circ f$. By monotonicity of $\succsim_u$, $e \succsim_u u^f \succsim_u -e$. By continuity and monotonicity of $\succsim_u$, there exists unique $\beta \in [0, 1]$, such that $\beta e + (1 - \beta)(-e) \sim_u u^f$. Hence, by Lemma 8, $(2\beta - 1)e \sim_u u^f$ if and only if $\beta z + (1 - \beta)w \sim f$. Thus $\beta z + (1 - \beta)w$ is a certainty equivalent of f. $\square$

Turning our attention to complementary pairs of acts, notice that if the pair of acts (f, g) is a complementary pair then $\frac{1}{2}U \circ f + \frac{1}{2}U \circ g = ke$ for some $k \in [-1, 1]$. Results equivalent to the next two lemmas appear in Siniscalchi [23] but we provide independent proofs. First we show that the induced preferences satisfy the following property.

Definition 12 *(Additivity for complementary pairs).* For any two pairs of complementary utility vectors $(u, \bar{u})$ and $(u'\bar{u}')$ in $[-1, 1]^n \times [-1, 1]^n$: if $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$ then $\lambda u + \gamma u' \sim_u \lambda\bar{u} + \gamma\bar{u}'$ for all $\lambda, \gamma \geqslant 0$, such that $\lambda u + \gamma u'$ and $\lambda\bar{u} + \gamma\bar{u}'$ are both in $[-1, 1]^n$.

Lemma 10 *(Additivity for complementary pairs). The induced preferences $\succsim_u$ exhibit the property 'additivity for complementary pairs' if and only if the underlying preferences $\succsim$ satisfy, axiom* A.5, *complementary independence.*

Proof. (*Sufficiency*) Fix two pairs of complementary utility vectors $(u, \bar{u})$ & $(u'\bar{u}')$ in $[-1, 1]^n \times [-1, 1]^n$ and fix $\lambda, \gamma \geqslant 0$ such that $\lambda u + \gamma u'$ and $\lambda\bar{u} + \gamma\bar{u}'$ are both in $[-1, 1]^n$. Suppose $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$. Since the pairs are complementary, there exist k, k' in $[-2, 2]$, such that $u = k - \bar{u}$ and $u' = k' - \bar{u}'$.

So consider the four acts $f : S \to \Delta(\{w, z\})$, $\bar{f} : S \to \Delta(\{w, z\})$, $g : S \to \Delta(\{w, z\})$ and $\bar{g} : S \to \Delta(\{w, z\})$, satisfying $U \circ f = (\lambda + \gamma)u$, $U \circ \bar{f} = (\lambda + \gamma)\bar{u}$, $U \circ g = (\lambda + \gamma)u'$ and $U \circ g = (\lambda + \gamma)\bar{u}'$. Notice the pairs $(f, \bar{f})$ and $(g, \bar{g})$ are complementary since, $\frac{1}{2}f + \frac{1}{2}\bar{f} = \frac{([\lambda+\gamma]k)+1}{2}z + \frac{(-[\lambda+\gamma]k)+1}{2}w$ and similarly, $\frac{1}{2}g + \frac{1}{2}\bar{g} = \frac{([\lambda+\gamma]k')+1}{2}z + \frac{(-[\lambda+\gamma]k')+1}{2}$.

Now let x_0 be the constant act for which $U(x_0) = 0$.

We first show that $f \sim \bar{f}$ and $g \sim \bar{g}$.

Case (i). $\lambda + \gamma \geqslant 1$. Suppose $f \sim \bar{f}$ fails, and in particular, suppose, without loss of generality, that $f \succ \bar{f}$. By continuity there exist $\varepsilon > 0$, and an act $\hat{f}$, satisfying $U \circ \hat{f} = U \circ f - \varepsilon e$ and $\hat{f} \sim \bar{f}$. Notice that by construction, $\hat{f}$ is complementary with $\bar{f}$. Since (x_0, x_0) is trivially a complementary pair with $x_0 \sim x_0$, applying A.5 (complementary independence) we have

$$\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0 \sim \frac{1}{\lambda + \gamma}\bar{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0.$$

But since

$$U \circ \left[\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0\right] = u - \varepsilon e, \quad \text{and}$$

$$U \circ \left[\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0\right] = \bar{u},$$

this implies, $u - \varepsilon e \sim_u \bar{u} \sim_u u$, contradicting the monotonicity of $\succsim_u$.

Case (ii). $\lambda + \gamma < 1$. Let $(f'', \bar{f}'')$ be the complementary pair of acts for which $U \circ f'' = u$ and $U \circ \bar{f}'' = \bar{u}$. Hence $f'' \sim \bar{f}''$. Recall (x_0, x_0) is trivially a complementary pair with $x_0 \sim x_0$, so by applying A.5, complementary independence, we have

$$(\lambda + \gamma) f'' + (1 - \lambda - \gamma) x_0 \sim (\lambda + \gamma) \bar{f}'' + (1 - \lambda - \gamma) x_0.$$

But since $U \circ [(\lambda + \gamma) f'' + (1 - \lambda - \gamma) x_0] = U \circ f = u$ and $U \circ [(\lambda + \gamma) \bar{f}'' + (1 - \lambda - \gamma) x_0] = U \circ \bar{f} = \bar{u}$, $u \sim_u \bar{u}$ implies $f \sim \bar{f}$. Similarly, it follows $g \sim \bar{g}$.

Applying complementary independence to $(f, \bar{f})$ and $(g, \bar{g})$ for $\alpha = \lambda/(\lambda + \gamma)$ yields $\alpha f + (1 - \alpha) g \sim \alpha \bar{f} + (1 - \alpha)\bar{g}$. And since

$$U \circ \big(\alpha f + (1 - \alpha) g\big) = \frac{\lambda}{\lambda + \gamma} U \circ f + \frac{\gamma}{\lambda + \gamma} U \circ g = \lambda u + \gamma u' \quad \text{and}$$
$$U \circ \big(\alpha \bar{f} + (1 - \alpha) \bar{g}\big) = \frac{\lambda}{\lambda + \gamma} U \circ \bar{f} + \frac{\gamma}{\lambda + \gamma} U \circ \bar{g} = \lambda \bar{u} + \gamma \bar{u}',$$

we have $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$, as required. □

(*Necessity*) Fix a pair of complementary acts $(f, \bar{f})$, $(g, \bar{g})$ and α in $(0, 1)$. And set $u := U \circ f$, $\bar{u} := U \circ \bar{f}$, $u' := U \circ g$ and $\bar{u}' := U \circ \bar{g}$. Suppose $u \succsim_u \bar{u}$ and $u' \succsim_u \bar{u}'$ then if $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$ by additivity for complementary pairs $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$ for $\lambda = \alpha$ and $\gamma = (1 - \alpha)$. Hence for the underlying preferences we have: $\alpha f + (1 - \alpha) g \sim \alpha \bar{f} + (1 - \alpha) \bar{g}$, as required.

If either of the preferences are strict, for example, say $u \succ_u \bar{u}$ (and $\lambda > 0$), then by monotonicity and continuity of $\succsim$ there exist $\hat{f}$ and $\varepsilon > 0$, such that $U \circ \hat{f} = u - \varepsilon e \sim_u \bar{u}$. By construction $\hat{f}$ is complementary to $\bar{f}$, so by additivity for complementary pairs we have $\lambda(u - \varepsilon e) + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$. Hence, by monotonicity of $\succsim_u$, it follows that $\lambda u + \gamma u' \succ_u \lambda \bar{u} + \gamma \bar{u}'$. Thus for the underlying preferences we have: $\alpha f + (1 - \alpha) g \succ \alpha \bar{f} + (1 - \alpha) \bar{g}$, as required. □

With this property in hand, we can now construct the baseline measure π to rank complementary state-utility vectors.

Lemma 11 *(Base-line measure). Assume* A.5. *Then there exists a unique $\pi \in \Delta(S)$, such that for all pairs of state-utility vectors u and u' in $[-1, 1]^n$, u and u' have a common mean μe if and only if $\mu = \mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$.*

Proof. To aid in this construction, it is convenient to introduce the following notation. Recall $e = (1, \dots, 1)$. Let e^s denote the unit vector with a 1 in the sth position and zeros elsewhere. Thus for any vector $u \in \mathbb{R}^n$ we have $u = \sum_{s=1}^{n} u_s e^s$.

For any u in $[-1, 1]^n$, we shall say any vector $\bar{u}$ in $\mathbb{R}^n$ is complementary to u, if $\bar{u} = ke - u$, for some $k \in \mathbb{R}$.

We will construct the π state-by-state by finding for each unit vector e^s, a λ^s in $(0, 1]$ and a complementary vector to which $\lambda^s e^s$ is indifferent. These two vectors will have the same mean (according to the baseline measure) and their 'difference-from-the-mean' vectors will be the negative complement of each other. But of course the mean of $\lambda^s e^s$ according to the baseline measure will be $\lambda^s \pi_s$ and that is how we can recover π_s.

Fix s in S. For each $\hat{\lambda}$ in $[0, 1]$, set $\bar{U}^s_{\hat{\lambda}} := \{u' \in \mathbb{R}^n \colon u' = ke - \hat{\lambda} e^s\}$, that is, the set of vectors that are complementary to $\hat{\lambda} e^s$. Since, by monotonicity, $\hat{\lambda} e^s \succsim_u -\hat{\lambda} e^s$ the set $\{u' \in [-1, 1]^n \colon \hat{\lambda} e^s \succsim_u u'\} \cap \bar{U}^s_{\hat{\lambda}}$ is also non-empty. Note also that $\bar{U}^s_0$ is the constant utility line so

$\{u' \in [-1,1]^n\colon u' \succsim_u 0e^s\} \cap \bar{U}^s_{\hat{\lambda}}$ is non-empty. But it may be the case that for sufficiently large $\hat{\lambda}$ that $\{u' \in [-1,1]^n\colon u' \succsim_u \hat{\lambda}e^s\} \cap \bar{U}^s_{\hat{\lambda}}$ is empty. So we shall set λ^s equal to the maximum $\hat{\lambda}$ in $[0,1]$ for which $\{u' \in [-1,1]^n\colon u' \succsim_u \hat{\lambda}e^s\} \cap \bar{U}^s_{\hat{\lambda}}$ is non-empty. It follows from the continuity and monotonicity of $\succsim_u$ that there is a unique $k^s \in [0,2]$, such that $\lambda^s k^s e - \lambda^s e^s \sim_u \lambda^s e^s$, so set $\pi_s := k^s/2$. By monotonicity of $\succsim_u$, it follows that $\pi_s \in [0,1]$. Moreover, by axiom A.7 (common-mean certainty independence), $\lambda k^s e - \lambda e^s \sim_u \lambda e^s$ for all $\lambda \in (0, \lambda^s]$. So set $\underline{\lambda} := \min_s \lambda^s$. Thus for each s:

$$\big(\underline{\lambda}2\pi_s e - \underline{\lambda}e^s\big) \sim_u \underline{\lambda}e^s, \quad \text{and} \quad \frac{1}{2}\big(\underline{\lambda}2\pi_s e - \underline{\lambda}e^s\big) + \frac{1}{2}\underline{\lambda}e^s = \underline{\lambda}\pi_s e.$$

To check that $\mathrm{E}_\pi(e) = 1$, we can apply the 'additivity for complementary pairs' $n-1$ times, by adding the pairs $\underline{\lambda}2\pi_s e - \underline{\lambda}e^s$ to obtain:

$$\begin{aligned}
&\sum_{s=1}^n \big(\underline{\lambda}2\pi_s e - \underline{\lambda}e^s\big) \sim_u \sum_{s=1}^n \underline{\lambda}e^s \\
&\quad\Rightarrow\quad \underline{\lambda}\left[\sum_{s=1}^n (2\pi_s - 1)\right] e \sim_u \underline{\lambda}e.
\end{aligned}$$

So by monotonicity it follows $2\sum_{s=1}^n \pi_s - 1 = 1 \Rightarrow \sum_{s=1}^n \pi_s = 1$, as required.

To obtain the mean of any vector u in $[-1,1]^n$, let $k^u \in [-2,2]$ be the unique number for which $\underline{\lambda}k^u e - \underline{\lambda}u \sim_u \underline{\lambda}u$. Now,

$$\underline{\lambda}k^u e - \underline{\lambda}u = \sum_{s=1}^n \underline{\lambda}\big(k^u - u_s\big)e^s \sim_u \sum_{s=1}^n \underline{\lambda}u_s e^s = \underline{\lambda}u.$$

Recall that, for each s, $(\underline{\lambda}2\pi_s e - \underline{\lambda}e^s, \underline{\lambda}e^s)$ is a complementary pair and the two state-utility vectors are indifferent to each other. Applying additivity for complementary pairs $n-1$ times we obtain:

$$\begin{aligned}
&s\sum_{s=1}^n u_s\underline{\lambda}\big(2\pi_s e - e^s\big) \sim_u \sum_{s=1}^n u_s\underline{\lambda}e^s \\
&\quad\Rightarrow\quad 2\mu\underline{\lambda}e - \underline{\lambda}u \sim_u \underline{\lambda}u,
\end{aligned}$$

where $\mu = \mathrm{E}_\pi(u)$. That is, we have shown that $\mu = k^u/2$.

Now suppose $(u, \bar{u})$, $(u', \bar{u}')$ are two complementary pairs of state-utility vectors satisfying $\underline{\lambda}u \sim \underline{\lambda}\bar{u}$ and $\underline{\lambda}u' \sim \underline{\lambda}\bar{u}'$. From the argument above it follows that $\bar{u} = 2(\mathrm{E}_\pi(u))e - u$ and $\bar{u}' = 2(\mathrm{E}_\pi(u'))e - u'$. From the definition of common mean, $\underline{\lambda}u$ and $\underline{\lambda}u'$ share a common mean only if

$$\frac{1}{2}\underline{\lambda}u + \frac{1}{2}\underline{\lambda}\bar{u} = \frac{1}{2}\underline{\lambda}u' + \frac{1}{2}\underline{\lambda}\bar{u}'.$$

The LHS is equal to

$$\underline{\lambda}\left[\frac{1}{2}u + \frac{1}{2}\big(2\big(\mathrm{E}_\pi(u)\big)e - u\big)\right] = \underline{\lambda}\big(\mathrm{E}_\pi(u)\big)e,$$

and the RHS is equal to

$$\underline{\lambda}\left[\frac{1}{2}u' + \frac{1}{2}\left(2\left(\mathrm{E}_\pi\left(u'\right)\right)e - u'\right)\right] = \underline{\lambda}\left(\mathrm{E}_\pi\left(u'\right)\right)e.$$

Hence equating coefficients, $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$, as required. □

The next three lemmas show that $\succsim_u$ satisfies convexity, radial homotheticity and translation invariance on common-mean sets, respectively. For all three lemmas let $U(\cdot)$ be an affine representation of $\succsim$ on X for which $U(w) = -1$ and $U(z) = 1$. Let π be the unique base-measure from Lemma 11. For a fixed μ in $[-1, 1]$, suppose that x^μ is a mean for all acts f such that $\mathrm{E}_\pi(U \circ f) = \mu$.

Lemma 12 *(Common-mean convexity). Assume* A.5 (*complementary independence*) *and* A.6 (*common-mean uncertainty aversion*) *apply. The restriction of $\succsim_u$ to state-utility vectors u in $[-1, 1]^n$ such that $\mathrm{E}_\pi(u) = \mu$ is convex.*

Proof. Fix u and u' in $[-1, 1]^n$ such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u') = \mu$ and suppose wlog $u \succsim_u u'$. We need to show $\alpha u + (1-\alpha)u' \succsim_u u'$, for all α in $(0, 1)$. Since $u \succsim_u u'$ it follows from Lemma 8 that there exist two acts $f : S \to \Delta(\{w, z\})$ and $g : S \to \Delta(\{w, z\})$ with $U \circ f = u$, $U \circ g = u'$ and $f \succsim g$. Since f and g share a common mean we can apply A.6 (common-mean uncertainty aversion) to obtain $\alpha f + (1-\alpha)g \succsim f$. Hence $U \circ (\alpha f + (1-\alpha)g) = \alpha u + (1-\alpha)u' \succsim_u u'$, as required. □

Lemma 13 *(Common-mean radial homotheticity). Assume* A.5 (*complementary independence*) *and* A.7 (*certainty invariance*) *apply. Then for all u' and u'' in $[-1, 1]^n$, such that $\mathrm{E}_\pi(u') = \mathrm{E}_\pi(u'') = \mu$ and all $\alpha \in (0, 1)$, $u' \succsim_u u''$ if and only if $\alpha u' + (1-\alpha)\mu e \succsim_u \alpha u'' + (1-\alpha)\mu e$.*

Proof. By definition $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u') = \mu$ and $u \succsim_u u'$ if and only if there exists a pair of acts f and g, such that $U \circ f = u$, $U \circ g = u'$ and $f \succsim g$. Fix an α in $(0, 1)$ and set $x^\mu := \frac{(\mu+1)}{2}[z] + \frac{(-\mu+1)}{2}[w]$. Notice that $U(x^\mu) = \mu$ and so x^μ is a common mean for f and g. Applying A.7 (certainty invariance) we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)x^\mu \succsim \alpha g + (1-\alpha)x^\mu$. And since $U \circ (\alpha f + (1-\alpha)x^\mu) = \alpha u' + (1-\alpha)\mu e$ and $U \circ (\alpha g + (1-\alpha)x^\mu) = \alpha u'' + (1-\alpha)\mu e$, we have $u \succsim_u u'$ if and only if $\alpha u' + (1-\alpha)\mu e \succsim_u \alpha u'' + (1-\alpha)\mu e$, as required. □

Lemma 14 *(Common-mean translation invariance). Assume* A.5, *and* A.7 *apply. Then for all u and u' in $[-1, 1]^n$, such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u') = \mu$ and all $\delta \in \mathbb{R}$ such that $u + \delta e$ and $u' + \delta e$ are both in $[-1, 1]^n$, $u \succsim_u u'$ if and only if $u + \delta e \succsim_u u' + \delta e$.*

Proof. Fix u and u' in $[-1, 1]^n$, such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u') = \mu$ and, without loss of generality, consider $\delta > 0$ such that $u + \lambda e$ and $u' + \delta e$ are both in $[-1, 1]^n$. Consider the vectors $v = \lambda^{-1}(u + e) - e$, and $v' = \lambda^{-1}(u' + e) - e$, where $\lambda = (2 - \delta)/2$ (< 1). Notice that

$$\lambda v + (1-\lambda)(-e) = u + e - \lambda e - (1-\lambda)e = u,$$

$$\lambda v + (1-\lambda)(e) = u + e - \lambda e + (1-\lambda)e = u + 2(1-\lambda)e = u + \delta e,$$

$$\lambda v' + (1-\lambda)(-e) = u',$$

$$\lambda v + (1-\lambda)(e) = u' + \delta e.$$

Since v lies on the ray from $(-e)$ that goes through u as well as lying on the ray from (e) that goes through $u+\delta e$, it follows that v is in $[-1,1]^n$. Similarly for v'. Thus there exists a pair of acts $f: S \to \Delta(\{w,z\})$ and $g: S \to \Delta(\{w,z\})$, such that $U \circ f = v$, $U \circ g = v'$, and, from Lemma 8, $v \succsim_u v'$ if and only if $f \succsim g$. As $\mathrm{E}_\pi(v) = \mathrm{E}_\pi(v') = \lambda^{-1}(\mu+1) - 1$, it follows that f and g have a common mean, namely, the constant act:

$$\left(\frac{\mu+1}{2\lambda}\right)[z] + \left(1 - \frac{\mu+1}{2\lambda}\right)[w],$$

which has utility $\lambda^{-1}(\mu+1) - 1$. Thus, applying certainty invariance twice, we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)w \succsim \alpha g + (1-\alpha)w$ if and only if $\alpha f + (1-\alpha)z \succsim \alpha g + (1-\alpha)z$. But by construction:

$$\begin{aligned}
&U\big(\lambda f + (1-\lambda)w\big) = \lambda v + (1-\lambda)(-e) = u,\\
&U\big(\lambda g + (1-\lambda)w\big) = \lambda v' + (1-\lambda)(-e) = u',\\
&U\big(\lambda f + (1-\lambda)w\big) = \lambda v + (1-\lambda)e = u + \delta e,\\
&U\big(\lambda f + (1-\lambda)w\big) = \lambda v' + (1-\lambda)e = u' + \delta e.
\end{aligned}$$

Hence $u \succsim_u u'$ if and only if $u + \delta e \succsim_u u' + \delta e$, as required. $\square$

Construction of representation. Let $U(.)$ be an affine representation of $\succsim$ on X for which $U(w) = -1$ and $U(z) = 1$. By Lemma 9, we know that, for all acts f in $\mathcal{F}$, there exists a constant act $x(f) \sim f$. Set $V(f) := U(x(f))$ to be the representation of $\succsim$ on $\mathcal{F}$. The corresponding representation for the induced preferences $\succsim_u$ over state utilities u' in $[-1,1]^n$ is $W(u') := U(x(f))$ for all f such that $U \circ f = u'$. By Lemma 8, it is enough to show that we can write this representation in the form $W(u') = \varphi(\mathrm{E}_\pi(u'), \rho(u'))$ where π is the unique base-measure from Lemma 11 and where the functions φ and ρ have the properties stated in Theorem 2.

Let π be the base measure of Lemma 11. For $\mu \in \mathbb{R}$, let H_π^μ be the hyperplane $\{u \in \mathbb{R}^n\colon\ \mathrm{E}_\pi(u) = \mu\}$. Suppose the induced preferences $\succsim_u$ are such that for all $\mu \in [-1,1]$, $u \in H_\pi^\mu \cap [-1,1]^n$ implies $u \sim \mu e$. In this case, the preferences are subjective expected utility and have representation $V(f) = \mathrm{E}_\pi(U \circ f)$, which is the trivial special case of our representation with $\varphi(\mu,\rho) \equiv \mu$. So instead assume that there exist a $\mu \in [-1,1]$ and a $\hat{u} \in H_\pi^\mu \cap [-1,1]^n$ such that $\mu e \succ_u \hat{u}$.

For all $u \in H_\pi^\mu \cap [-1,1]^n$, the vector $\alpha(u - \mu e)$ lies in $H_\pi^0 \cap [-1,1]^n$ for α sufficiently small. By Lemma 13, since $\mu e \succ_u \hat{u}$, we have $\mu e \succ_u \alpha\hat{u} + (1-\alpha)\mu e$ for all $\alpha \in (0,1]$, and by Lemma 14, $\mu e \succ_u \alpha\hat{u} + (1-\alpha)\mu e$ if and only if $0 \succ_u \alpha(\hat{u} - \mu e)$ for $\alpha(\hat{u} - \mu e) \in [-1,1]^n$. Hence it is without loss of generality to take $\mu = 0$ and $\hat{u} \in H_\pi^0 \cap [-1,1]^n$ such that $0e \succ_u \hat{u}$.

The preference relation $\succsim_u$ confined to $H_\pi^0 \cap [-1,1]^n$ is homothetic, hence we can extend it to the whole of H_π^0 as follows. For all u in H_π^0 let $\bar{\beta}(u) = \max\{\beta \in (0,1]\colon\ \beta u \in [-1,1]^n\}$. Now, with slight abuse of notation, for all u, u' in H_π^0, let $u \succsim_u u'$ if and only if $\beta u \succsim_u \beta u'$ for all $\beta \in (0, \min\{\bar{\beta}(u), \bar{\beta}(u')\})$. The extended relation inherits common mean convexity.

Since this preference relation on H_π^0 is homothetic with $0e$ as a bliss point, it admits a linear homogeneous representation, $-\rho$, with $\rho(0) = 0$ where $\rho : H_\pi^0 \to \mathbb{R}$ is unique up to positive scalar multiplication. To normalize, let $\rho(\hat{u}) = -W(\hat{u})$; that is, $\rho(\hat{u}) := U(x(f))$ for any f such that $U \circ f = \hat{u}$. Common mean convexity of $\succsim_u$ and linear homogeneity of ρ imply that ρ is subadditive on H_π^0.

We now extend the dispersion measure ρ from H_π^0 to the whole of $\mathbb{R}^n$ (and in particular to the whole of $[-1,1]^n$) by setting $\rho(u) := \rho(u - \mathrm{E}_\pi(u)e)$ for any u in $\mathbb{R}^n$. By construction, ρ is translation invariant, and so the extended function inherits linear homogeneity and subadditivity from its restriction to H_π^0. For all u, u' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$, we have $\rho(u) \geqslant \rho(u')$ if and only if $u' \succsim_u u$. To see this, fix u, u' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u') = \mu$. Then $\rho(u) \geqslant \rho(u')$ if and only if $u' - \mu e \succsim_u u - \mu e$ (in the extended relation on H_π^0) if and only if $\beta(u' - \mu e) \succsim_u \beta(u - \mu e)$ for $\beta \in (0, \min\{\bar\beta(u - \mu e), \bar\beta(u' - \mu e)\})$ (in the original relation) if and only if $\beta(u') \succsim_u \beta(u)$ if and only if $u' \succsim_u u$.

Also, by construction, $\rho(\mu e) = 0$ for all $\mu \in [-1,1]$, and for all complementary pairs $(u, \bar u)$ with common mean (that is, such that $\frac{1}{2}u + \frac{1}{2}\bar u = \mathrm{E}_\pi(u)e = \mathrm{E}_\pi(\bar u)e$) we have $u \sim \bar u$ which implies $\rho(u) = \rho(\bar u)$. By the translation invariance of ρ, this means that for any complementary pair $(u, \bar u)$, with or without a common mean, $\rho(u) = \rho(\bar u)$. In particular, $\rho(u) = \rho(-\bar u)$.

Define $\varphi(\mu, \rho') := W(u')$ for all u' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u') = \mu$ and $\rho(u') = \rho'$. This is well-defined since for all u', u'' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u') = \mathrm{E}_\pi(u'') = \mu$ and $\rho(u') = \rho(u'')$, by the definition of ρ, we have $u' \sim_u u''$.

The function is increasing in its first argument and satisfies $\varphi(\mu, 0) = \mu$ by construction. It is also non-increasing in its second argument since we showed above that for all u, u' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$, we have $\rho(u) \geqslant \rho(u')$ if and only if $u' \succsim_u u$. $\square$

(*Necessity of axioms*) For a mean-dispersion representation $\langle U, \pi, \rho, \varphi\rangle$ with bounded U, and with φ and ρ satisfying

$$\left[\varphi\left(\mathrm{E}_\pi(u), \rho\left(u - \left(\mathrm{E}_\pi(u)\right)e\right)\right) - \varphi\left(\mathrm{E}_\pi\left(u'\right), \rho\left(u' - \left(\mathrm{E}_\pi\left(u'\right)\right)e\right)\right)\right] \cdot \left(u - u'\right) \geqslant 0$$

for all $u, u' \in U(X)^n$, it is immediate that the associated preferences over acts satisfy axioms A.1–A.3.

Lemma 15 shows that invariant symmetric preferences satisfy A.6 (common-mean uncertainty aversion).

Lemma 15 *(Common-mean uncertainty aversion). Fix a bounded invariant symmetric representation $\langle U, \pi, \rho, \varphi\rangle$. The associated preferences over acts $\succsim$ satisfy: for any two acts f and g in $\mathcal{F}$, if $f \sim g$ then $\alpha f + (1-\alpha)g \succsim f$.*

Proof. Given $\langle U, \pi, \rho, \varphi\rangle$, f and g having a common mean and $f \sim g$ implies $\varphi(\mu, \rho(u)) = \varphi(\mu, \rho(u'))$, where $u = U \circ f$, $u' = U \circ g$ and $\mu = \mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$. Hence $\rho(u) = \rho(u')$,

$$\begin{aligned}
V\left(\alpha f + (1-\alpha)g\right) &= \varphi\left(\mu, \rho\left(\lambda u + (1-\lambda)u'\right)\right) \\
&\geqslant \varphi\left(\mu, \lambda\rho(u) + (1-\lambda)\rho\left(u'\right)\right) \\
&= \varphi\left(\mu, \rho(u)\right) = V(f).
\end{aligned}$$

The inequality follows from the convexity of ρ (which holds since ρ is both positive linearly homogeneous and sub-additive) and that φ is non-increasing in its second argument. $\square$

Lemma 16 shows that invariant symmetric preferences satisfy A.7 (certainty invariance).

Lemma 16 *(Certainty invariance). Fix an invariant symmetric representation $\langle U, \pi, \rho, \varphi\rangle$. The associated preferences over acts $\succsim$ satisfy: for any two acts f and g in $\mathcal{F}$, such that $\mathrm{E}_\pi(U \circ f) = \mathrm{E}_\pi(U \circ g)$, any constant act x and any α in $(0,1)$, $f \succsim g \Leftrightarrow \alpha f(1-\alpha)x \succsim \alpha g + (1-\alpha)x$.*

Proof. Given $\langle U, \pi, \rho, \varphi\rangle$, it is immediate that the induced preferences $\succsim_u$ over state-utility vectors satisfy common-mean translation invariance: in fact, for any pair of utility vectors u and

u' in $\mathbb{R}^n$, s.t. $\mathrm{E}_\pi(u) = \mathrm{E}_\pi(u')$ and any $\delta \in \mathbb{R}$, $u \succsim_u u' \Rightarrow u + \delta e \succsim_u u' + \delta e$. Fix acts f and g in $\mathcal{F}$, such that $\mathrm{E}_\pi(U \circ f) = \mathrm{E}_\pi(U \circ g)$ and constant acts x and y in X and α in $(0, 1)$. Set $\delta := (1-\alpha)(U \circ y(s) - U \circ x(s))$ and notice that

$$\begin{aligned}
&\big[\alpha U \circ f + (1-\alpha)U \circ y\big] - \big[\alpha U \circ f + (1-\alpha)U \circ x\big] \\
&\quad = \big[\alpha U \circ g + (1-\alpha)U \circ y\big] - \big[\alpha U \circ g + (1-\alpha)U \circ x\big] \\
&\quad = \delta e.
\end{aligned}$$

Hence $\alpha U \circ f + (1-\alpha)U \circ x \succsim_u \alpha U \circ g + (1-\alpha)U \circ x$ implies $\alpha U \circ f + (1-\alpha)U \circ y \succsim_u \alpha U \circ g + (1-\alpha)U \circ y$. Therefore $\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$ implies $\alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y$, as required. □

Our final lemma shows that invariant symmetric preferences exhibit additivity for complementary pairs.

Lemma 17 *(Additivity for complementary pairs II). Fix an invariant symmetric representation $\langle U, \pi, \rho, \varphi\rangle$. The associated preferences over state-utilities $\succsim_u$ exhibit the property of 'additivity for complementary pairs.*

Proof. Take any two pairs of complementary state-utility vectors $(u, \bar{u})$ and $(u', \bar{u}')$ and any $\lambda, \gamma \geqslant 0$. If $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$ then we know from Lemma 11 that $\frac{1}{2}u + \frac{1}{2}\bar{u} = \mu e$ and $\frac{1}{2}u' + \frac{1}{2}\bar{u}' = \mu' e$, where $\mu = \mathrm{E}_\pi(u)$ and $\mu' = \mathrm{E}_\pi(u')$. Hence, $\mathrm{E}_\pi(\bar{u}) = \mu$, $\mathrm{E}_\pi(\bar{u}') = \mu'$.

Furthermore, $\mathrm{E}_\pi(\lambda u + \gamma u') = \mathrm{E}_\pi(\lambda \bar{u} + \gamma \bar{u}') = \lambda\mu + \gamma\mu'$, and

$$\frac{1}{2}\big(\lambda u + \gamma u'\big) + \frac{1}{2}\big(\lambda \bar{u} + \gamma \bar{u}'\big) = \big(\lambda\mu + \gamma\mu'\big)e.$$

So it follows from complementary symmetry of ρ, that $\rho(\lambda u + \gamma u') = \rho(\lambda \bar{u} + \gamma \bar{u}')$.

$$\begin{aligned}
W\big(\lambda \bar{u} + \gamma \bar{u}'\big) &= \varphi\big(\lambda\bar{\mu} + \gamma\bar{\mu}', \rho\big(\lambda \bar{u} + \gamma \bar{u}'\big)\big) \\
&= \varphi\big(\lambda\mu + \gamma\mu', \rho\big(\lambda u + \gamma u'\big)\big) \\
&= W\big(\lambda u + \gamma u'\big),
\end{aligned}$$

as required. □

Finally, Lemma 10 demonstrates the necessity of Axiom A.5 (complementary independence) for $\succsim_u$ to exhibit the property of additivity for complementary pairs. This completes the proof of necessity. □

Proof of Proposition 3. Fix acts f and g in $\mathcal{F}$, a constant act y in X and an α in $(0, 1)$. Suppose $f \trianglerighteq g$. That is, there exist a constant act x, and a λ in $[0, 1]$, such that $\lambda f + (1-\lambda)x = g$. We have

$$\begin{aligned}
\lambda\big(\alpha f + (1-\alpha)y\big) + (1-\lambda)\big(\alpha x + (1-\alpha)y\big) &= \alpha\big(\lambda f + (1-\lambda)x\big) + (1-\alpha)y \\
&= \alpha g + (1-\alpha)y.
\end{aligned}$$

Hence, $\alpha f + (1-\alpha)y \trianglerighteq \alpha g + (1-\alpha)y$.

Alternatively, suppose $\alpha f + (1-\alpha)y \trianglerighteq \alpha g + (1-\alpha)y$. By the definition of $\trianglerighteq$, this entails that there exist a constant act x and a λ in $[0, 1]$, such that

$$\lambda\big(\alpha f + (1-\alpha)y\big) + (1-\lambda)\big(\alpha x + (1-\alpha)y\big) = \alpha g + (1-\alpha)y.$$

Thus,

$$\alpha\big(\lambda f + (1-\lambda)x\big) + (1-\alpha)y = \alpha g + (1-\alpha)y.$$

Equating parts of each side of this equation that are multiplied by α yields $\lambda f + (1-\lambda)x = g$. That is, $f \trianglerighteq g$, as required. $\square$

Proof of Proposition 4 (Decreasing absolute uncertainty aversion). Part I. We first establish the following two preliminary results.

I.1. $f \trianglerighteq g \Rightarrow \rho(U \circ f) \geqslant \rho(U \circ g)$.

I.2. For any $\hat{\rho} \in \rho([-1,1]^n)$, and any $0 \leqslant \rho' < \hat{\rho}$, there exist acts f and g for which $\rho(U \circ f) = \hat{\rho}$, $\rho(U \circ g) = \rho'$ and $f \trianglerighteq g$.

Proof of I.1. As $f \trianglerighteq g$, there exist a constant act x and $\lambda \in [0,1]$, such that $g = \lambda f + (1-\lambda)x$. Hence $\rho(U \circ g) = \rho(U \circ [\alpha f + (1-\alpha)x]) = \alpha\rho(U \circ f) \leqslant \rho(U \circ f)$ (by the translation invariance and homogeneity of $\rho(\cdot)$). $\square$

Proof of I.2. To see this, fix a $\hat{\rho}$ for which $\rho(\hat{u}) = \hat{\rho}$, for some $\hat{u}$ in $[-1,1]^n$. The act f in which $f_s = (\frac{1+\hat{u}_s}{2})[b] + (\frac{1-\hat{u}_s}{2})[w]$, is an act that, by construction, is one in which $U \circ f = \hat{u}$ and hence its measure of dispersion $\rho(U \circ f) = \rho(\hat{u}) = \hat{\rho}$. Furthermore, $\bar{f}$, where $\bar{f}_s = (\frac{1-\hat{u}_s}{2})[b] + (\frac{1+\hat{u}_s}{2})[w]$ is, by construction, complementary to f, since $\frac{1}{2}f + \frac{1}{2}\bar{f} = \frac{1}{2}[b] + \frac{1}{2}[w]$, a constant act. Finally, for the act

$$g = \frac{\rho'}{\hat{\rho}} f + \left(1 - \frac{\rho'}{\hat{\rho}}\right)\left(\frac{1}{2}[b] + \frac{1}{2}[w]\right),$$

the measure of dispersion of its associated state-utility vector $\rho(U \circ g) = \rho(\frac{\rho'}{\hat{\rho}}\hat{u}) = \frac{\rho'}{\hat{\rho}} \times \rho(\hat{u}) = \rho'$ (by the homogeneity of $\rho(.)$), and $f \trianglerighteq g$, as required. $\square$

Part II. Proof of "(a) implies (b)".

In terms of the induced preferences $\succsim_u$ over state-utility vectors, DAUA requires that if $\hat{u} \trianglerighteq \hat{v}$ then, for any $d < d'$ and any α,

$$\alpha\hat{u} + (1-\alpha)de \sim_u \alpha\hat{v} + (1-\alpha)de \quad \Rightarrow \quad \alpha\hat{u} + (1-\alpha)d'e \succsim_u \alpha\hat{v} + (1-\alpha)d'e$$

or setting $u := \alpha\hat{u} + (1-\alpha)de$, $v := \alpha\hat{v} + (1-\alpha)de$, and $\delta := (1-\alpha)(d'-d) > 0$

$$u \sim_u v \quad \Rightarrow \quad u + \delta e \succsim_u v + \delta e. \tag{10}$$

Notice from the translation invariance of $\rho(\cdot)$, $\rho(u + \delta e) = \rho(u)$ and $\rho(v + \delta e) = \rho(v)$. By Proposition 3 it follows that $u \trianglerighteq v$, so the implication (10) holds whenever the associated vectors are in $[-1,1]$, that is, are in the domain of $\succsim_u$.

Now suppose $\varphi(\mu, \rho) = \varphi(\mu', \rho')$ with $\mu > \mu'$. Since φ is increasing in its first argument and non-increasing in its second, it follows that $\rho > \rho'$. And from result I.2, we have established that there exists a u in $[-1,1]^n$, such that $\mathrm{E}_\pi(u) = \mu$ and $\rho(u) = \rho$. Since $\mu' < \mu$ and $\rho' < \rho$, it follows there exists a u' in $[-1,1]^n$ such that $\mathrm{E}_\pi(u') = \mu'$ and $\rho(u') = \rho'$. Applying (10) yields $\varphi(\mu + \delta, \rho) \geqslant \varphi(\mu' + \delta, \rho')$, as required. $\square$

Part III. Proof of "(b) implies (a)".

Fix a pair of acts f, g in $\mathcal{F}$ such that $f \trianglerighteq g$, a pair of constant acts $y \succsim x$, and $\alpha \in (0,1)$. Set $u := \alpha U \circ f + (1-a)U(x)e$, $v := \alpha U \circ g + (1-a)U(x)e$, $\delta := (1-\alpha)(U(y) - U(x))$.

Suppose $\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$, that is, $\varphi(\mathrm{E}_\pi(u), \rho(u)) \geqslant \varphi(\mathrm{E}_\pi(v), \rho(v))$.

By Proposition 3, $\alpha f + (1-\alpha)x \trianglerighteq \alpha g + (1-\alpha)x$, and applying result I.1 we have $\rho(\hat{u}) > \rho(v)$, and hence $\mathrm{E}_\pi(\hat{u}) > \mathrm{E}_\pi(v)$. Now from the continuity and monotonicity of φ with respect to its first argument, there exists $k \geqslant 0$, such that for $\hat{u} := u - ke$, $\varphi(\mathrm{E}_\pi(\hat{u}), \rho(\hat{u})) = \varphi(\mathrm{E}_\pi(v), \rho(v))$.

Applying condition (b), in conjunction with the monotonicity of φ with respect to its first argument and the translation invariance of $\rho(\cdot)$, yields:

$$\begin{aligned}\varphi\big(\mathrm{E}_\pi(u+\delta e), \rho(u+\delta e)\big) &\geqslant \varphi\big(\mathrm{E}_\pi(\hat{u})+\delta, \rho(\hat{u})\big) \geqslant \varphi\big(\mathrm{E}_\pi(v)+\delta, \rho(v)\big) \\ &= \varphi\big(\mathrm{E}_\pi(v+\delta e), \rho(v+\delta e)\big).\end{aligned}$$

And

$$\begin{aligned}&\varphi\big(\mathrm{E}_\pi(u+\delta e), \rho(u+\delta e)\big) \geqslant \varphi\big(\mathrm{E}_\pi(v+\delta e), \rho(v+\delta e)\big) \\ &\quad \Rightarrow \quad u+\delta e \succsim_u v+\delta e.\end{aligned}$$

But by construction: $u+\delta e = \alpha U \circ f + (1-a)U(y)e$ and $v + \delta e = \alpha U \circ g + (1-a)U(y)e$. Hence,

$$\alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y,$$

as required. $\square$

Part IV. To show the implication for smooth preferences, consider a path along the indifference curve from (μ, ρ) to (μ', ρ'), with slope $\partial\rho/\partial\mu$ given at any point by $-\varphi_1/\varphi_2$. Condition (b) holds if and only if φ_1 is non-decreasing along this path. The change in φ_1 for a small movement along the path is given by $\varphi_{11} + \varphi_{12}\frac{\partial\rho}{\partial\mu}$. Hence we require

$$\begin{aligned}&\varphi_{11} + \frac{\varphi_1}{-\varphi_2}\varphi_{12} \geqslant 0 \\ &\quad \Rightarrow \quad \frac{\varphi_{12}}{-\varphi_2} \geqslant \frac{-\varphi_{11}}{\varphi_1},\end{aligned}$$

as desired. $\square$

Proof of Corollary 5. (i) $\Leftrightarrow$ (iii). From Proposition 4(b) CAUA holds if and only if $\varphi(\mu + \delta, \bar{\rho}) = \varphi(\mu, \bar{\rho}) + \delta$ for all μ, δ, ρ satisfying the stated conditions which is true if and only if (ii) holds. $\square$

Proof of Corollary 6. (a) is a special case of Proposition 4, part (b), and (b) follows immediately from (a). $\square$

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