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Generalized neo-additive capacities and updating

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This paper shows that, for Choquet expected utility preferences, the axioms *consequentialism, state independence* and *conditional certainty equivalent consistency* under updating characterise a family of capacities, which we call *Generalized Neo-Additive Capacities* (GNAC). This family contains as special cases, among others, neo-additive capacities as introduced by Chateauneuf, Eichberger, and Grant, Hurwicz capacities, and ε-contaminations. Moreover, we will show that the convex version of a GNAC is the only capacity for which the core of the full Bayesian updates of a capacity, introduced by Jaffray, equals the set of Bayesian updates of the probability distributions in the core of the original capacity.

Key words ambiguity, Choquet Expected Utility, consequentialism, dynamic consistency updating

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1 Introduction

A major problem when modelling ambiguity of a decision-maker in a dynamic context lies in the well-known precarious relationship between updating capacities or multiple priors, dynamic consistency and consequentialism. Early work by Epstein and LeBreton (1993) and Eichberger and Kelsey (1996) showed that updating Choquet expected utility (CEU) preferences, which satisfy consequentialism, in a dynamically consistent way implies additive beliefs. Even if dynamic consistency was constrained to an event tree, ambiguous beliefs modelled by a capacity were possible only on the final partition of events (Sarin and Wakker 1998; Eichberger, Grant, and Kelsey 2005). For ambiguity models with multiple priors, Epstein and Schneider (2003) found that the set of priors had to fulfill a fairly restrictive rectangularity condition in order to represent dynamically consistent preferences. In particular, the original Ellsberg paradox cannot be explained with rectangular sets of priors.

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We would like to dedicate this article to Jean-Yves Jaffray. His contributions to the theory of decision-making under ambiguity have greatly influenced our work. His personal communications and comments have provided great encouragement for us.

In the light of these results, there are essentially two ways to proceed. Either one can abandon consequentialism and all the models relying on it, like CEU and multiple priors, or give up dynamic consistency. The former route has been explored by Hanany and Klibanoff (2007). In this paper, we retain consequentialism.

In the spirit of Gilboa and Schmeidler (1993), we consider a preference relation and the family of its updated preferences which satisfy the two axioms *Consequentialism* and *State Independence*. For the case where beliefs can be described by multiple priors, as in the max–min expected utility (MEU) preference model of Gilboa and Schmeidler (1989), Pires (2002) proved that these two axioms plus a third axiom that Eichberger, Grant, and Kelsey (2007) refer to as *Conditional Certainty Equivalent Consistency* (CCEC) are equivalent to the full Bayesian updating of all prior probabilities. If the preference relation can be represented by a Choquet integral and beliefs by a capacity, as in the CEU preference model of Schmeidler (1989), Eichberger, Grant, and Kelsey (2007) and Horie (2007) established that *Consequentialism*, *State Independence*, and a weakening of CCEC that Horie refers to as *Conditional Certainty Equivalent Consistency for Binary Acts*, are equivalent to full Bayesian updating of the capacity as suggested by Jaffray (1992) and Walley (1991).[1]

In this paper, we characterize the family of capacities for which the three initial axioms of Pires (2002) hold; that is, where CCEC is not restricted to binary acts. We find a class of capacities which is slightly more general than the family of neo-additive capacities which were introduced and axiomatized in Chateauneuf, Eichberger, and Grant (2007). For CEU preferences with an associated neo-additive capacity, the CEU of an act can be expressed as a convex combination of the expected utility with respect to an additive probability distribution and the Hurwicz criterion (Hurwicz 1951) which itself is a convex combination of the utility values of the best and the worst outcomes.

The CEU of an act with respect to generalized neo-additive capacities (GNAC) combines the subjective expected utility of the act and the utility values of the best and the worst outcomes linearly, but the combinations need no longer be convex but merely monotone. In addition, we can show that convex GNACs are the only capacities for which the core of the full Bayesian update of a capacity coincides with the set of Bayesian updates of the probabilities in the core of the original capacity. These results provides a further justification for neo-additive capacities as a useful restriction on the CEU approach.[2]

The paper is organized as follows. After introducing the basic framework in the next section, we prove in Section3 that CEU preferences satisfy Axiom CCEC if and only if the capacity ν is a GNAC. Section4 discusses several special cases of GNACs before concluding. Unless otherwise stated the proofs are gathered in the Appendix.

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[1] Horie (2007) showed that the necessary conditions in Eichberger, Grant, and Kelsey (2005) were too stong and suggested the appropriate weakening of the CCEC axiom where CCEC is restricted to hold only for binary acts.

[2] In a recent paper, Chateauneuf, Gajdos, and Jaffray (2010) investigate robust updating rules. From the perspective of robust statistics introduced by Huber (1981), they show that updated (ε, δ)-*contaminations* are the only class of capacities whose core coincides with the set of Bayesian updates of the probabilities in the core of the original capacity. (ε, δ)-*contaminations* are the special case of convex GNACs. In this regard, proposition 2 of Chateauneuf, Gajdos, and Jaffray (2010) overlaps with Proposition 4 in this paper.

2 The framework

Let S be a finite set of states of the world, $\Sigma = \mathcal{P}(S)$, the set of events in S. For any $E \in \Sigma$, let E^c denote the complement of E. Let X be a set of outcomes. An *act* is a function $f : S \to X$, and $\mathcal{F}$ denotes the set of such acts. Given any two acts f and g in $\mathcal{F}$ and any event E in Σ, we denote by $f_E g \in \mathcal{F}$ the act defined as $f_E g(s) = f(s)$ if $s \in E$ and $f_E g(s) = g(s)$ otherwise. For notational convenience, we will not distinguish between the outcome $x \in X$ and the constant act $x \in \mathcal{F}$, defined as $x(s) = x$ for all $s \in S$.

Binary acts are an important special case. For any two outcomes x and y in X and any event E in Σ, the binary act $x_E y$ is defined as

$$x_E y(s) = \begin{cases} x & \text{if } s \in E \\ y & \text{otherwise.} \end{cases}$$

A capacity ν is a set function from Σ to $\mathbb{R}$ with $\nu(\emptyset) = 0, \nu(S) = 1$ and $\nu(A) \leq \nu(B)$ for all $A \subset B$, A and B in Σ.

Given a von Neumann–Morgenstern utility function $u : X \to R$, two acts f and g are comonotonic with respect to u, if for all pairs of states s and s' in S,

$$[u(f(s)) - u(f(s'))][u(g(s)) - u(g(s'))] \geq 0.$$

The CEU of an act f with respect to the capacity ν is given by

$$CEU(f, \nu) = \int_{-\infty}^{0} (\nu(u(f(s)) \geq t) - 1)dt + \int_{0}^{+\infty} \nu(u(f(s)) \geq t)dt.$$

Since acts are finite-valued they can be written as $f = \sum_{i=1}^{n} x_i A_i$, where, in a convenient abuse of notation, we denote by A_i both the set $A_i \in \Sigma$ and the indicator function of the set A_i. That is, $A_i(s) = 1$ for $s \in A_i$ and 0 otherwise. For any $f \in \mathcal{F}$, we denote by $[f]$ the subset of acts comonotonic to f, which are measurable with respect to the partition $A_1, \ldots A_n$. Without loss of generality, suppose that the finite outcomes $x_i \in f(S)$ are ordered such that $u(x_i) \leq u(x_{i+1})$, then

$$\begin{aligned} CEU(f, \nu) &= \sum_{i=1}^{n} u(x_i) \cdot [\nu(A_i \cup A_{i+1} \cup \cdots \cup A_n) - \nu(A_{i+1} \cup A_{i+2} \cup \cdots \cup A_n)] \\ &= \sum_{i=1}^{n} u(x_i) \cdot m_{[f]}(A_i), \end{aligned}$$

with $m_{[f]}(A_i) := [\nu(A_i \cup A_{i+1}..A_n) - \nu(A_{i+1} \cup A_{i+2}..A_n)]$. Note that $\sum_{i=1}^{n} m_{[f]}(A_i) = 1$ holds. Thus, one can view the Choquet integral as determined by a set of probability distributions m, one for each possible ordering of outcomes.

Throughout this paper, we will consider preference relations $\succsim$ on F which can be represented by a CEU functional,

$$f \succsim g \iff CEU(f, \nu) \geq CEU(g, \nu).$$

Conditional CEU preferences $\succsim_E$ are defined by conditional capacities ν_E.

2.1 Null events

If the outcomes of any act on some event can be changed in an arbitrary way without affecting its evaluation by the decision-maker, such an event was called a null event by Savage (1954). For modelling ambiguity by CEU preferences, this notion of a null set is too strong. Following Ghirardato and Marinacci (2002), an event $E \in \Sigma$ is null (universal) if $x_E y \sim y$ ($x_E y \sim x$) for all pairs of outcomes $x, y \in X$ with $x \succ y$. An event E is essential if for some $x, y \in X$, $x \succ x_E y \succ y$. We denote by $\mathcal{N}$ the sets of such null events. If preferences $\succsim$ are represented by a CEU functional, then $\nu(E) = 0$ if and only if $E \in \mathcal{N}$. Notice that CEU preferences do not imply that the complement of a null event is universal.

In our analysis below, we do not restrict capacities to have non-zero values for events other than the empty set. This generality requires us to impose the following axiom and to introduce additional regularity conditions.

Axiom 0 (*Null-Event Consistency*)
For all pairs of outcomes x and y such that $x \succ y$, $x_E y \sim y$ implies $y_E x \sim x$.

Even for CEU preferences it is not true in general that the union of two null sets will be a null set. The following definition characterizes sets of events which satisfy some regularity conditions which will be implied by our axioms on updating below.

Definition 1 (*Ideal*) *A collection of sets* $\mathcal{T} \subset \Sigma$ *is called an* ideal *if*
(i) $\varnothing \in \mathcal{S}$ *and* $S \notin \mathcal{T}$*;*
(ii) $A \in \mathcal{T}$ *implies* $B \in \mathcal{T}$ *for all* $B \subset A$*;*
(iii) $A \in \mathcal{T}$ *and* $B \in \mathcal{T}$ *implies* $A \cup B \in \mathcal{T}$.

Note that properties (i) and (ii) are satisfied by the set $\{E \in \Sigma |\ \nu(E) = 0\}$ for any capacity ν. This is not the case, however, for property (iii).[3]

Below, we want to consider classes of capacities which share the same set of null events $\mathcal{N}$ and for which the the set of null events is an ideal.

Definition 2 *Fix a collection of null events* $\mathcal{N}$*. A capacity* ν *is congruent with respect to* $\mathcal{N}$*, if* $A \in \mathcal{N} \Rightarrow \nu(A) = 0$ *and* $\nu(A^c) = 1$ (*or equivalently,* $\nu(A) > 0 \Rightarrow A \notin \mathcal{N}$).

A capacity is congruent with a collection of null events $\mathcal{N}$ if it assigns a capacity value of zero to every element in this collection. Equivalently, if an event has positive capacity value then it cannot be an element of $\mathcal{N}$. Capacities whose set of null events forms an ideal are known as *null-additive set functions* (Pap 1995).

2.2 Updating preferences

Consider a family of preference relations $\succsim_E$ on $\mathcal{F}$ which represents the decision-maker's preferences after it becomes known that the non-null event E has occurred. The ex-ante unconditional preference relation on $\mathcal{F}$ will be denoted by $\succsim$.

[3] We will see in Lemma 2.1 that property (iii) is a consequence of an axiom which will be introduced below.

For preferences which are additive, that is, represented by a CEU functional with an additive capacity ν, standard Bayesian updating satisfies the following three axioms.

Axiom SI (*State Independence*)
For any two outcomes $x, y \in X$, and any non-null event $E \notin \mathcal{N}$,

$$x \succsim y \Leftrightarrow x \succsim_E y.$$

Axiom C (*Consequentialism*)
For any two acts $f, g \in \mathcal{F}$, and any event $E \in \Sigma$,

$$\text{if } f = g \text{ on } E, \text{ then } f \sim_E g.$$

Axiom DC (*Dynamic Consistency*)
For any acts $f, g \in \mathcal{F}$ and any non-null event $E \in \Sigma$,

$$f \succ_E g \iff f_E g \succ g.$$

State independence requires the conditional preferences over outcomes to agree with the unconditional preferences over outcomes. Consequentialism rules out effects on future choices from outcomes which would have become relevant in the event E^c, which did not happen. Dynamic consistency links conditional and unconditional preferences by requiring that preferences after E occurred remain consistent with ex-ante preferences.

It is well known (Ghirardato 2002) that together the three axioms imply for a CEU decision-maker that the utility function remains unchanged for the conditional preferences and the capacity ν is additive and updated by Bayes rule. As we wish to retain the property that the ordinal ranking over outcomes is not affected by which event we condition upon, we will have to maintain state independence. Hence, for an updating rule that leaves room for uncertainty represented by a non-additive capacity we must relax either consequentialism or dynamic consistency.

Retaining consequentialism, Pires (2002) proposes a weaker version of DC, conditional certainty equivalent consistency, which restricts the act g of the classical DC axiom to be constant.

Axiom CCEC (*Conditional Certainty Equivalent Consistency*)
For any non-null event E, any outcome $x \in X$, and any act f in $\mathcal{F}$,

$$f \sim_E x \iff f_E x \sim x. \tag{1}$$

If the acts f in equation (1) are restricted to binary acts, then we refer to Axiom CCEC as *Conditional Certainty Equivalent Consistency for Binary Acts.*

Applying Axiom 0 reveals immediately the implication that the complement of a null event is universal. Although this appears to be a natural assumption if capacities are supposed to represent beliefs, this property is not implied by the CEU representation directly. Moreover, together with Axiom CCEC (see Lemma 1 below), Axiom 0 implies that the union of two null events must be also null; that is, that the set of null events are an ideal.

The following lemma[4] shows that the updating axiom, Axiom CCEC, in combination with Axiom 0 yields a set of null events which is closed under the union operation (property (iii) in the ideal definition).

Lemma 1 *If Axiom 0 and Axiom CCEC for binary acts hold, then* $\nu(A) = \nu(B) = 0$ *implies* $\nu(A \cup B) = 0$.

PROOF: Let A and B be such that $\nu(A) = \nu(B) = 0$ and $\nu(A \cup B) \neq 0$ holds. By Axiom 0, $\nu(A^c) = 1$. For $x, y, z \in X$ with $x \prec y \prec z$, let $f = xA + zB$ and, for $E = A \cup B$, let $f_E y = xA + yE^c + zB$. Then $\int f_E y d\nu = x(1 - \nu(A^c)) + y(\nu(A^c) - \nu(B)) + z\nu(B) = y$. Hence, $f_E y \sim y$. The same reasoning holds for any $y' \in X$ with $y' \neq y$, $x \prec y' \prec z$. Hence, $f_E y' \sim y'$. By Axiom CCEC, $f \sim_E y \sim_E y'$ for all $y' \in X$, $x \prec y' \prec z$. W.l.o.g. assume $x \prec y \prec y' \prec z$. By Axiom CCEC, $y' \sim_E y$ if and only if $y'_E y \sim y$. Hence, $\int \left(y'_E y\right) d\nu = y(1 - \nu(E)) + y'\nu(E)) = y$ and $\nu(E) = \nu(A \cup B) \neq 0$ imply $y \sim y'$, a contradiction. □

In the context of ambiguity, the conclusion of Lemma 1 appears less innoxious. After all, ambiguity may manifest itself in a decision-maker's inability to assign probability values to subevents, even if he or she feels capable of such a judgment for the union of the events. This implication, however, is driven by Axiom CCEC, and hence has to be judged in context with the latter.

There are several well-known updating rules for capacities in the literature. In Eichberger, Grant, and Kelsey (2010), some of these have been compared. Among them, the *Full Bayesian Updating* (FBU) rule of Jaffray (1992) and Walley (1991) proved to be particularly suitable for maintaining a distinction between ambiguity and ambiguity attitude in a dynamic context.

Definition 3 *Given a non-null event* E, *the FBU rule is defined as the capacity* ν_E,

$$\nu_E(A) := \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(A \cup E^c)}.$$

As the next section will show, the FBU rule is closely related to Axiom CCEC.

3 Generalized neo-additive capacities (GNACs)

For multiple-prior preferences, Pires (2002) proved that state independence, consequentialism and conditional certainty equivalent consistency imply the FBU rule, where each probability distribution in the set of priors is updated according to Bayes' rule. In the CEU context, conditional and unconditional CEU preferences satisfy Axiom SI and Axiom C by definition. Moreover, from Eichberger, Grant, and Kelsey (2007) and Horie (2007) we know

[4] Note that the proof uses only binary act consistency, the weaker notion of Axiom CCEC which was suggested by Horie (2007). Notice that Axiom 0 would also have to be assumed for the main result in Horie (2007), if it were extended to null sets other than the empty set.

that Axiom CCEC restricted to binary acts implies that the capacities of CEU preferences are updated according to the FBU rule suggested by Jaffray (1992) and Walley (1991).

We will show now that Axiom 0 and Axiom CCEC in its full strength determine a class of capacities similar to neo-additive capacities, as axiomatized in Chateauneuf, Eichberger, and Grant (2007). We will call these capacities Generalized Neo-Additive Capacities (GNACs).

A GNAC is defined as a linear affine transformation of an additive probability distribution which satisfies the monotonicity condition. It is a slight generalization of neo-additve capacities of Chateauneuf, Eichberger, and Grant (2007) where the capacities do not need to be convex combinations of the probability and an Hurwicz capacity.

Definition 4 *Let $\mathcal{N}$ be a collection of null events which is an ideal, π a finitely additive probability measure on (S, Σ), and a and $b > 0$ a pair of numbers satisfying* $\min_{E\notin\mathcal{N}}[a + b\pi(E)] \geq 0$ *and* $\max_{E\notin\mathcal{N}}[a + b(1 - \pi(E))] \leq 1$, *then a Generalized Neo-Additive Capacity $\nu(\cdot|\mathcal{N}, \pi, a, b)$ is defined as*

$$\nu(E|\mathcal{N}, \pi, a, b) := \begin{cases} 0 & \text{if} \quad E \in \mathcal{N} \\ a + b\pi(E) & \text{if} \quad E \notin \mathcal{N} \quad \text{and} \quad E^c \notin \mathcal{N} \\ 1 & \text{if} \quad E^c \in \mathcal{N} \end{cases}.$$

The Choquet expected value of an act f with respect to the GNAC $\nu(E|\mathcal{N}, \pi, a, b)$ is easily computed as

$$\begin{aligned} CEU(f, \nu) = b \int_{\{s, \exists t\in S, (u\circ f)(s)\leq(u\circ f)(t), \nu(\{t\})\neq 0\}} (u \circ f) d\pi \\ + a \cdot \max\{x|\ x \in (u \circ f)(\{s, \nu(\{s\}) \neq 0\})\} \\ + (1 - a - b) \cdot \min\{x|\ x \in (u \circ f)(\{s, \nu(\{s\}) \neq 0\})\}). \end{aligned}$$

In Chateauneuf, Eichberger, and Grant (2007) the following property of a capacity was introduced.

Property A $\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G)$ is satisfied for all events E, F and G such that $\nu(F) \neq 0$, $\nu(F \cup E) \neq 1$, $\nu(G) \neq 0$ and $\nu(G \cup E) \neq 1$.

This property implies that the capacity ν has an additive part. It is an easy exercise to check that a GNAC satisfies Property A.

We now state our main result.

Proposition 1 *Consider a CEU preference relation $\succsim$ with utility index u and capacity ν and a collection of CEU preferences $(\succsim_E)_{E\in\Sigma}$ with capacities $(\nu_E)_{E\in\Sigma}$ which satisfy Axiom 0, Axiom SI, and Axiom C. Then the following statements about the preference order $\succsim$ and its updates $\succsim_E$ are equivalent:*

(i) Axiom CCEC is satisfied.
(ii) ν is updated according to FBU and Property A is satisfied.
(iii) ν is updated according to FBU and ν is a GNAC.

The relationship between the axioms and the representation result in Proposition 1 is quite subtle. By its representation, CEU preferences satisfy Axiom SI and Axiom C. As argued before, CEU preferences do not imply Axiom 0. By CCEC these properties are transferred to the CEU updates of the unconditional preference order. Given these properties:

- *CCEC for binary acts* is equivalent to the capacities being updated by FBU (see Eichberger, Grant, and Kelsey (2007) and Horie (2007));
- *CCEC*, in contrast to CCEC for binary acts, is equivalent to FBU updating *and* Property A; and
- Property A is essentially equivalent to the capacities belonging to the family of GNACs with the same ideal of null-events.

Similarly to Epstein and LeBreton (1993) and Ghirardato (2002), who provide axioms that imply Bayesian updating together with SEU, in Proposition 1 we show that Choquet preferences satisfying CCEC, and not merely the CCEC for binary acts, imply FBU updating *and* a special form of the capacity which we call GNAC.

The following remark indicates that small generalizations of the result in Proposition 1 are possible.

Remark 1 *Possible generalizations:*

(i) It is worth noting that our proof uses only one version of Axiom CCEC, namely $f \sim_E x \Rightarrow f_E x \sim x$.

(ii) In the statement of Axiom CCEC, we could replace the constant act x by a slightly more general act:

Alternative Axiom: *Suppose* $\varnothing \neq \arg_{s\in S}\min(u\circ f)(s) \cap \arg_{s\in S}\min(u\circ g)(s) \subset A$ *and* $\varnothing \neq \arg_{s\in S}\max(u\circ f)(s) \cap \arg_{s\in S}\max(u\circ g)(s) \subset A$, *then for any* $h \in \mathcal{F}$ *such that*

$$\max\left\{\min_{s\in S}(u\circ f)(s), \min_{s\in S}(u\circ g)(s)\right\} \leq \min_{s\in S}(u\circ h)(s),$$
$$\max_{s\in S^c}(u\circ h)(s) \leq \min\left\{\max_{s\in S}(u\circ f)(s), \max_{s\in S}(u\circ g)(s)\right\}.$$

$$f \sim_A g \quad \text{if and only if} \quad f_A h \sim g_A h.$$

This alternative axiom is stronger than CCEC, but for CEU preferences it is equivalent to CCEC. Hence, for CEU preferences, Axiom CCEC implies GNAC which in turn implies the alternative axiom.

3.1 Neo-additive capacities and GNACs

Non-extreme-outcome–additive capacities with null sets in $\mathcal{N}$ were introduced and axiomatized by Chateauneuf, Eichberger, and Grant (2007). These capacities are additive except for events which have extreme outcomes. They are defined by a probability π, which is congruent with $\mathcal{N}$, and by the parameters $\alpha, \delta \in [0, 1]$,

$$\nu(E|\mathcal{N}, \pi, \delta, \alpha) = \begin{cases} 0 & \text{if} \quad E \in \mathcal{N} \\ (1-\delta)\pi(E) + \delta\alpha & \text{if} \quad E \notin \mathcal{N} \text{ and } E^c \notin \mathcal{N} \\ 1 & \text{if} \quad E^c \in \mathcal{N} \end{cases}$$

Neo-additive capacities are GNACs with $a := \delta\alpha \geq 0$, $b := (1-\delta) \leq 1$, and an additive probability distribution π which is congruent with $\mathcal{N}$.

Both neo-additive capacities and GNACs are linear affine transformations of an additive probability distribution. For a GNAC, however, the probability distribution π need not be congruent with the set of null sets $\mathcal{N}$ of the capacity ν. For a GNAC with additive probability distribution π, it is possible that there exists an event E such that $\pi(E) > 0$ and $\nu(E) = 0$ holds. Moreover, for any essential event F, such that $E \cap F = \varnothing$, $\nu(F \cup E) = a + b[\pi(F) + \pi(E)] = \nu(F) + b\pi(E) > \nu(F)$. Hence, for a GNAC it is possible that a null event has an impact on the evaluation of an act even if the act does not associate an extreme outcome with it.

For neo-additive capacities, π must be congruent with the set of null events $\mathcal{N}$. Therefore, for neo-additive capacities, a null event can affect the evaluation of an act only if an extreme outcome occurs on it. Whether this is a desirable property of a capacity depends on the application. If capacities are supposed to model beliefs, experimental results may be required in order to decide which axiom in respect to null events is more appropriate.

In order to impose congruence between a GNAC ν and its additive part π (i.e., $\pi(E) = 0$ implies $\nu(E) = 0$) a strengthening of Axiom 0 is necessary.

Axiom 0′ (*Strong Null-Event Consistency*)

For any three outcomes $x, y, z \in X$ with $x \succ y \succ z$, any null event $N \in \mathcal{N}$ and any event $E \in \Sigma$, such that $E \cap N = \emptyset$,

$$y_N(x_E z) \sim x_E z. \tag{2}$$

Strong null-event consistency requires a null event not to affect the evaluation of a bet as long as it carries a non-extreme outcome.[5] For $E = \varnothing$, equation (2) coincides with the definition of a null event.

Lemma 2 below shows that Axiom 0′ implies that the capacity of a CEU preference order is a null-additive set function in the sense of Pap (1995).

Lemma 2 *CEU preferences satisfy Axiom 0′ if and only if for any null event $N \in \mathcal{N}$ and any event $E \in \Sigma$,* such that $E \cap N = \emptyset$, *one has*

$$\nu(E \cup N) = \nu(E).$$

[5] The property imposed by Axiom 0′ is implied by axiom 5 in Chateauneuf, Eichberger, and Grant (2007).

PROOF: Applying CEU to equation (2) one has

$$\begin{aligned} &u(x)\nu(E) + u(y)[\nu(E \cup N) - \nu(E)] + u(z)[1 - \nu(E \cup N)] \\ &\quad = u(x)\nu(E) + u(z)[1 - \nu(E)] \\ &\quad \Longleftrightarrow [u(y) - u(z)][\nu(E \cup N) - \nu(E)] = 0. \end{aligned}$$ □

For a GNAC, $\nu(E \cup N) = \nu(E)$ implies $\pi(N) = 0$. Hence, Axiom 0′ does induce consistency of π with the null events in $\mathcal{N}$.

The Choquet integral of GNACs satisfying Axiom 0′ is a convex combination of the expected utility of an act with respect to π and the max and min over the support of π.[6] If Axiom 0′ and $a \geq 0$ holds, then the GNAC is a neo-additive capacity.

A natural question concerns the relationship between the system of axioms advanced for a GNAC in this paper and the system of axioms for neo-additive capacities in Chateauneuf, Eichberger, and Grant (2007). GNACs are axiomatized by axioms relating conditional and unconditional CEU preferences. In Chateauneuf, Eichberger, and Grant (2007), neo-additive capacities are characterized by restricting independence to acts which have the best and worst outcomes on the same set of states. It is not obvious how to relate these axioms because they apply to different frameworks. A careful and detailed study of this issue is beyond the scope of this paper.

4 GNACs and updating

Assuming CEU preferences which satisfy null-event consistency and consequentialism, Axioms 0 and C, updating rules impose conditions on unconditional preferences which may characterize these preferences. Thus, dynamic consistency implies additive capacities. Similarly, as the previous section has shown, CCEC implies that capacities are linear affine transformations of an additive probability distribution. Several well-known examples of capacities can be obtained as special cases of GNACs by putting additional constraints on the parameters a and b.

1. Subjective expected utility (Savage 1954): $a = 0$, $b = 1$.
2. Simple capacities, $\varepsilon -$ contaminations (Huber 1981): $a = 0$, $b \leq 1$.
3. $(\varepsilon, \delta) -$ *contaminations* (Chateauneuf, Gajdos, and Jaffray 2010): $-1 \leq a \leq 0$, $b < 1$.
4. Hurwicz capacity (Hurwicz 1951): $0 \leq a \leq 1$, $b = 0$.
4. Convex capacity: $a \leq 0$, $a + b \leq 1$.
5. Concave capacity: $a \geq 0$, $a + b \geq 1$.
6. Cavex capacity (Wakker 2001): $a \geq 0$, $a + b \leq 1$.

The updating axiom CCEC, however, also has implications for the updated preferences. In this section, we will derive and discuss some properties for the updates of GNACs.

[6] With the strong null event consistency Axiom 0′, GNACs also satisfy Property D in the Appendix. Therefore, Lemma 4 can be applied and the probability π is also congruent with $\mathcal{N}$.

Properties of the FBUs depend on the classification according to $a \gtreqless 0$ and $a + b \gtreqless 1$.

	$a + b \leq 1$	$a + b \geq 1$
$a \leq 0$	*convex*	*'vexcave'*
$a \geq 0$	*cavex*	*concave*

4.1 Convex GNACs ($a \leq 0$, $a + b \leq 1$)

Let ν be a convex capacity, and let $C(\nu) = \{p \in \Delta(S)|\ \ p \geq \nu\}$ denote its core. It is well-known (Schmeidler 1989) that $CEU(f, \nu) = \min_{p \in C(\nu)} \int (u \circ f)\, dp$ in this case. In the light of the result by Pires (2002) that an axiom like CCEC applied to a multiple-priors model yields that the set of priors after updating equals the set of the Bayesian updates of the priors, one may be inclined to think that a similar result would hold for CEU preferences. As Horie (2007) points out, however, and as one easily checks, we have $C(\nu)_E \subseteq C(\nu_E)$, where $C(\nu)_E$ denote the set of Bayesian updates with respect to E of the probabilities in the core $C(\nu)$. To see this notice that for each $p \in C(\nu)$ and any event E such that $p(E) > 0$,

$$\frac{p(A)}{p(E)} - \nu_E(A) = \frac{p(A)}{p(E)} - \frac{\nu(A)}{\nu(A) + \overline{\nu}(A^c \cap E)} = \frac{p(A)\overline{\nu}(E \backslash A) - \nu(A)\, p(E \backslash A)}{p(E)(\nu(A) + \overline{\nu}(E \backslash A))}$$

where $\overline{\nu}(A) := 1 - \nu(A^c)$ denotes the dual capacity of ν. As $p \in C(\nu)$, we have $p(A) \geq \nu(A)$ and $p(E \backslash A) \leq \overline{\nu}(E \backslash A)$. Hence, $p(A)/p(E) \geq \nu_E(A)$. Horie (2007) also shows by example that, in general, $C(\nu)_E \neq C(\nu_E)$.

The reverse inclusion $C(\nu_E) \subseteq C(\nu)_E$ holds, however, if the capacity is a convex GNAC and if there are at least four states.

Proposition 2 *If* $|S| > 3$, *then* $C(\nu)_E = C(\nu_E)$ *if and only if* ν *is a* convex *GNAC.*

An immediate consequence of Proposition 2 is the representation of the conditional preferences by the same type of functional. Since the FBU ν_E of a convex capacity ν is also convex,[7] it follows for a convex GNAC that

$$CEU(f, \nu_E) = \min_{p \in C(\nu)_E} \int (u \circ f)\, dp.$$

A couple of remarks may help to put Proposition 2 into the context of related work in the literature:

(i) Studying robust updating rules, Chateauneuf, Gajdos, and Jaffray (2010) derive the result of Proposition 2 for (δ, ε) -contaminations.
(ii) For a finite state space S, there exist convex GNACs which are not ε-contaminations. For example, $|S| = 4$ and $\pi(E) = \frac{|E|}{|S|}$, then $\nu = \frac{6}{5}\pi - \frac{1}{5}$ is convex, but not an ε-

[7] For a proof see, e.g., Chateauneuf and Jaffray (1995).

contamination. With a non-atomic state space S, however, monotonicity implies that the only convex GNACs are ε-contaminations.

(iii) Proposition 2 provides necessary and sufficient conditions for capacities to satisfy $C(\nu_E) = C(\nu)_E$. An alternative condition can be found in theorem 2 of Jaffray (1992). Proposition 2, however, holds for convex capacities whereas Jaffray's theorem 2 is true only for belief functions.

(iv) If $|S| = 3$ holds, then $C(\nu_E) = C(\nu)_E$ is true for every convex capacity.

(v) If $|S| > 3$, it follows from Proposition 1 that the only case in which $C(\nu_E) = C(\nu)_E$ holds for convex capacities is when Axiom CCEC is true.

Proposition 2 shows that GNACs are the only class of capacities for which the updated probabilities of the core coincide with the core of the FBUs of the capacity. This result can be generalized to arbitrary updating rules of convex capacities for which the core of the updated capacity coincides with the Bayesian updates of the probabilities in the core of the capacity.

Proposition 3 *Let P_E be the set of updates on event E of a set of priors P. If P_E is the core of a convex FBU ν_E of some capacity ν, then ν is a GNAC.*

Proposition 3 shows the strength of the consistency requirements for updating rules. The class of GNACs together with the FBU rule allow for a class of CEU representations with a reasonable degree of consistency in updating which covers a broad range of models used in economic applications.

For $a = 0$, convex GNACs coincide with the case of simple capacities or ε-contaminations which has been used extensively in the applied literature.[8] Following Schmeidler (1989), this case has been associated with ambiguity aversion or pessimism. For GNACs, however, the convex case allows also for $a < 0$. Monotonicity of the capacity imposes only the general constraint $\min_{E \notin \mathcal{N}} [a + b\pi(E)] \geq 0$. This constraint implies a lower bound for a, $a \geq -b \min_{E \notin \mathcal{N}} \pi(E)$ which equals 0 only if the probability distribution π puts probability 0 on some non-null event. The degree to which a can become negative depends on the probability distribution π. An intuitive interpretation in a decision context for $a < 0$, however, is not easy to find.

Remark 2 *For convex GNACs, the probability π will be congruent with $\mathcal{N}$. To see this, consider an event $N \in \mathcal{N}$ and an arbitrary event F with $F \cap N = \varnothing$. By convexity of ν, $\nu(N \cup F) + \nu(N^c) \leq \nu(S) + \nu(F)$. As $\nu(N^c) = 1$, one obtains $\nu(N \cup F) = \nu(F)$. Hence, by Lemma 2, the capacity satisfies Axiom 0′ and π is congruent with $\mathcal{N}$.*

4.2 Concave GNAC ($a \geq 0$, $a + b \geq 1$)

Concave GNACs are the duals of convex GNACs. Hence, it is not surprising that the results of the previous subsection hold mutatis mutandis. For a concave capacity ν, denote by $\bar{C}(\nu) = \{p \in \Delta(S)| \ p \leq \nu\}$ the *anti-core*. In this case, $CEU(f, \nu) = \max_{p \in \bar{C}(\nu)} \int (u \circ f)\, dp$.

[8] Examples of applications include Dow and Werlang (1992), Eichberger and Kelsey (2002), Eichberger and Kelsey (2004), Teitelbaum (2009), Jeleva and Rossignol (2008), and Eichberger, Kelsey, and Schipper (2009).

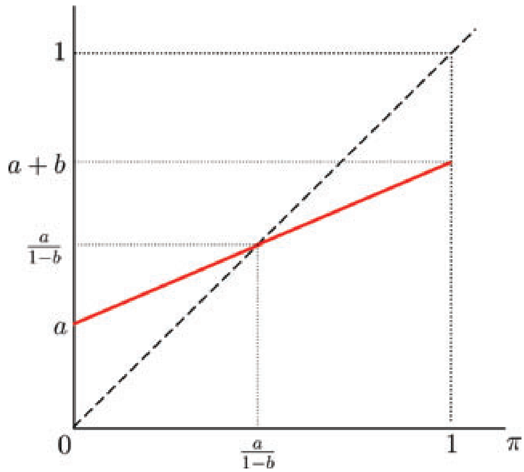

Figure 1 Updating cavex GNACs.

Given an event E with $p(E) > 0$, let $\bar{C}(\nu)_E$ be the set of Bayesian updates of the probabilities in the anti-core $\bar{C}(\nu)$. By analogous reasoning as in the previous case of convex capacities, we can show $\bar{C}(\nu)_E \subseteq \bar{C}(\nu_E)$. The conjugate of the capacity in example 1 of Horie (2007) illustrates that, in general, $\bar{C}(\nu)_E \neq C(\nu_E)$. Our next proposition, which we state without formal proof, shows that $\bar{C}(\nu)_E = \bar{C}(\nu_E)$ if and only if ν is a concave GNAC.

Proposition 4 *If* $|S| > 3$, *then* $\bar{C}(\nu)_E = \bar{C}(\nu_E)$ *if and only if* ν *is a* concave *GNAC.*

Hence, for a concave GNAC, we have as well the representation

$$CEU(f, \nu_E) = \max_{p \in \bar{C}(\nu)_E} \int (u \circ f)\, dp.$$

4.3 Cavex GNACs ($a \geq 0$, $a + b \leq 1$): Neo-additive capacities

Wakker (2001) introduced the notion of a *cavex capacity*: "Concavity is imposed on the unlikely events and convexity on the likely events. Henceforth, such capacities are called *cavex*" (p. 1049). Figure 1 shows the capacity value as a function of its additive part π.

A cavex GNAC is a neo-additive capacity as introduced and axiomatized by Chateauneuf, Eichberger, and Grant (2007).

Cavex GNACs can also be viewed as a special case of a type of capacity introduced by Jaffray and Philippe (1997). We will refer to such capacities as JP-capacities. A JP-capacity is a convex combination of a convex capacity and its conjugate, which is a concave capacity (i.e., $\nu := \alpha\mu + (1-\alpha)\overline{\mu}$ where μ is a convex capacity and $\alpha \in [0, 1]$).

A GNAC $\nu(.|\mathcal{N}, \pi, a, b)$ with the parameter restrictions $a \geq 0$ and $a + b \leq 1$ is a convex combination

$$\nu(\cdot|\mathcal{N}, \pi, a, b) = \frac{(1-a-b)}{(1-b)}\mu(\cdot|\mathcal{N}, \pi, b) + \frac{a}{(1-b)}\bar{\mu}(\cdot|\mathcal{N}, \pi, b).$$

of the convex capacity $\mu(E|\mathcal{N}, \pi, b)$,

$$\mu(E|\mathcal{N}, \pi, b) := \begin{cases} 0 & \text{if} \quad E \in \mathcal{N} \\ b\pi(E) & \text{if} \quad E \notin \mathcal{N} \text{and} \quad E^c \notin \mathcal{N} \\ 1 & \text{if} \quad E^c \in \mathcal{N}, \end{cases}$$

and its concave conjugate, defined by $\bar{\mu}(E|\mathcal{N}, \pi, b) = (1-b) + b\pi(E)$ for essential events $E, E^c \notin \mathcal{N}$.

Since μ is a convex capacity and $\bar{\mu}$ is a concave capacity, we can combine the arguments of the previous two subsections to obtain

$$CEU(f, \nu_E) = \frac{(1-a-b)}{(1-b)} \min_{p \in P_E} \int (u \circ f) dp + \frac{a}{(1-b)} \max_{p \in P_E} \int (u \circ f) dp, \tag{3}$$

where

$$P_E = C(\mu)_E = \bar{C}(\bar{\mu})_E.$$

A nice property of the FBU of a cavex GNAC is the fact that the weights given to the minimal and maximal expected utility in equation (3), ($(1-a-b)/(1-b)$ and $a/(1-b)$) are independent of the event on which the cavex GNAC is updated. In terms of Figure 1 the transformation of π will only change in slope but not in its fixed point.

4.4 'Vexcave' GNAC ($a \leq 0, a + b \geq 1$)

This parameter constellation of a GNAC is almost in contradiction with the monotonicity requirement of a capacity. In order to see this let $\underline{\pi} := \min_{E \notin \mathcal{N}} \pi(E)$ and consider the inequalities $\min_{E \notin \mathcal{N}} [a + b\pi(E)] = a + b\underline{\pi} \geq 0$ and $\max_{E \notin \mathcal{N}} [a + b(1 - \pi(E))] = a + b(1 - \underline{\pi}) \leq 1$. They imply $0 \geq a \geq -b\underline{\pi}$ and $1 \leq a + b \leq 1 + b\underline{\pi}$. For an atomless state space, these inequalities would force the GNAC to equal its additive part.

Though conceptually possible, this parameter constellation appears difficult to reconcile with observable attitudes towards ambiguity.

5 Conclusion

In this paper, we show that a decision-maker with CEU preferences satisfying *Consequentialism*, *State Independence*, and *Conditional Certainty Equivalent Consistency* will hold beliefs which are a linear transformation of an additive probability distribution. In a CEU model of decision-making under ambiguity, consistency requirements between unconditional and conditional preferences restrict the class of capacities considerably. The class of capacities determined by these three axioms almost coincides with neo-additive capacities. CEU preferences with neo-additive capacities can be represented as a linear combination of the expected utility with respect to some additive probability distribution and the maximum and minimum utility over outcomes.

These three axioms also imply that the capacity of a CEU preference order must be updated according to the FBU rule. If beliefs are represented by a convex capacity, then the core of the full Bayesian updated capacities equals the set of Bayesian updates of the probabilities in the core of the prior capacity. These observations clarify some open questions on fully Bayesian updating of capacities and multiple priors and provide additional arguments for generalized neo-additive capacities in a dynamic context.

Appendix

PROOFS: For the proof of Proposition 1 we will use the following two properties of capacities which characterize GNACs. These properties were introduced in proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007).[9]

Property A $\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G)$ is satisfied for all events E, F and G such that $\nu(F) \neq 0$, $\nu(F \cup E) \neq 1$, $\nu(G) \neq 0$ and $\nu(G \cup E) \neq 1$.

Property A characterizes capacities which have identical increments for essential events. For such capacities the Choquet integral will have the same additive probability distribution for all rank-orders of states with the same best and worst state.

Property D Let N be a null event and E an essential event, then $\nu(N \cup E) = \nu(E)$.

Property D characterizes capacities for which the union with a null event will not affect the capacity value of an essential event.

In the proof of Proposition 1 we will use the following characterization of GNACs.

Lemma 3 *Supposing that a capacity ν satisfies Property D, then the following assertions are equivalent: (i) ν is a GNAC, (ii) ν satisfies Property A.*

PROOF: We refer to the proof of proposition 3.1 in Chateauneuf, Eichberger, and Grant (2007), pp. 556–559. They prove that a capacity ν, which satisfies Properties A and D and two further properties, properties (b) and (c), is a null-additive set function of the form $\nu(E) = \lambda + (1 - \delta)\pi(E)$ for any essential set E, where π is a probability distribution on S and $\lambda, \delta \in [0, 1]$ are real numbers. A careful reading of their proof reveals that properties (b) and (c) are only used in part (b2) of their proof in order to establish $\lambda \leq 1$ and in part (c) in order to show that $\delta \in [0, 1]$. Therefore, if only Property A and Property D are satisfied, then the capacity has no bounds on λ and δ, except the ones implied by monotonicity. It follows that a capacity ν satisfying Properties A and D is a GNAC. □

Proof of Proposition 1: (i) We suppose that CCEC is fulfilled and show that Property A holds for a capacity ν satisfying Axioms 0 and CCEC.

[9] Property A corresponds to (a) and Property D to (d) in proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007).

For ease of notation we write $\int u(f)d\nu$ for $CEU(f,\nu)$ and $\int_E u(f)d\nu$ for $CEU(f\cdot E,\nu)$ where E is the indicator function of E. From theorem 1 in Eichberger, Grant, and Kelsey (2007), we obtain that the von Neumann–Morgenstern utility u can be chosen to be independent of E.

Step A: Let $f=\sum_{j=1}^{n}x_j A_j$ and $u(x_j)<u(x_{j+1})$ for all j, $1\le j\le n$. Note that there is an additive measure $m_{[f]}$ such that $\nu(A_j\cup A_{j+1}\cup\cdots\cup A_n)=m_{[f]}(A_j\cup A_{j+1}\cup\cdots\cup A_n)=\sum_{j=1}^{n}m_{[f]}(A_i)$ and $\int u(f)d\nu=\int u(f)dm_{[f]}$. For any event A_i, denote by $E_i:=A_i^c$. Then there is an (additive) measure $p_{[f]}$ such that $\nu_{E_i}(A_j\cup A_{j+1}\cup\cdots\cup A_n)=p_{[f]}(A_j\cup A_{j+1}\cup\cdots\cup A_n)=\sum_{j=1}^{n}p_{[f]}(A_i)$ for $j\ne i$, $1\le j\le n$, and $\int u(f)d\nu_{E_i}=\int u(f)dp_{[f]}$.

In this step we will show that for any event A_i, $i\ne 1,n$, on which the act f takes no extreme value, the Choquet integral of f conditional on the information that the event A_i did not occur can be calculated according to the measure $m_{[f]}$ updated by the event $E_i=A_i^c$ according to Bayes' rule $p_{[f]}=\frac{m_{[f]}}{m_{[f]}(E_i)}$.

Lemma 4 *Fix some act f. If ν satisfies Axiom CCEC, then for any A_i with $i\ne 1,n$, and associated measures $m_{[f]}$* and $p_{[f]}$ *such that $\int u(f)d\nu=\int u(f)dm_{[f]}$ and $\int u(f)d\nu_{E_i}=\int u(f)dp_{[f]}$, we have $p_{[f]}=\frac{m_{[f]}}{m_{[f]}(E_i)}$.*

Proof: Let y be the certainty equivalent of f conditional on E_i, $f\sim_{E_i}y$. If f and $f_{E_i}y$ are comonotone then $u(x_{i-1})<u(y)<u(x_{i+1})$. Otherwise, f and $f_{E_i}y$ are not comonotone, if $u(x_{i+1})<u(y)$ then we can decrease x_1 to $\widetilde{x}_1$ such that g defined by

$$g(s)=\begin{cases}\widetilde{x}_1 & \text{for} \quad s\in A_1\\ f(s) & \text{otherwise}\end{cases}$$

satisfies $g\sim_{E_i}y'$ and $u(x_{i-1})<u(y')<u(x_{i+1})$. Similarly, if $u(y)<u(x_{i-1})$, we can increase x_n to obtain g satisfying $g\sim_{E_i}y'$ and $u(x_{i-1})<u(y')<u(x_{i+1})$.

By construction, f and $g_{E_i}y'$ are comonotone. By axiom CCEC, $f_{E_i}y\sim y$ and $g_{E_i}y'\sim y'$.

Hence, $\int u(g)d\nu_{E_i}=\int u(g)dp_{[f]}$. As g and $g_{E_i}y$ are comonotone with f, their Choquet integrals are computed according to the same measure $m_{[f]}$, namely $\int u(g_{E_i}y')d\nu=\int u(g_{E_i}y')dm_{[f]}$. From $g_{E_i}y'\sim y'$, we get $u(y')=\int u(g_{E_i}y')d\nu=\int u(g_{E_i}y')dm_{[f]}=\int_{E_i}u(g)dm_{[f]}+m_{[f]}(A_i)u(y')$, which can be transformed to yield

$$u(y')=\frac{1}{m(E_i)}\int_{E_i}u(g)dm_{[f]}=\int_{E_i}u(g)d\nu_{E_i}=\int_{E_i}u(g)dp_{[f]}.$$

Let $\pi:=\frac{m_{[f]}}{m_{[f]}(E_i)}-p_{[f]}$, we have $\int_{E_i}u(g)d\pi=0$. In order to prove that $\pi=0$, for any $j\ne i$, define the act $g^{\epsilon,j}$ as follows:

$$g^{\epsilon,j}(s)=\begin{cases}x'_j & \text{for} \quad s\in A_j\\ g(s) & \text{otherwise}\end{cases}$$

Let $g^{\epsilon,j}\sim_{E_i}\widetilde{y}$. By continuity of u, we can chose x'_j such that $u(x'_j)=u(x_j)+\epsilon$ for some $\epsilon\in\mathbb{R}$. As $u(x_{i-1})<u(y')<u(x_{i+1})$,we can choose ϵ close enough to 0 such that $g^{\epsilon,j}$ and g are comonotone, and $g^{\epsilon,j}$ and $g^{\epsilon,j}_{E_i}\widetilde{y}$ are comonotone. By the same reasoning as before, one shows that $\int_{E_i}u(g^{\epsilon,j})d\pi=0$. As $\int_{E_i}u(g^{\epsilon,j})d\pi-\int_{E_i}u(g)d\pi=\epsilon\cdot\pi(A_j)=0$, we get for all $j\ne i$, $\pi(A_j)=0$ and therefore $p_{[f]}=\frac{m_{[f]}}{m_{[f]}(E_i)}$. □

Step B. From Lemma 4, we have $p_{[f]}=\frac{m_{[f]}}{m_{[f]}(E_i)}$ for each act f and each measure $m_{[f]}$ such that $\int u(f)d\nu=\int u(f)dm_{[f]}$ whenever there is no extreme outcome of f on A_i. Hence, by varying the best and worst outcomes, one can find two *comonotone* acts f and f' such that $x_i=f(A_i)$ and $x'_i=f'(A_i)$ satisfy $u(x_j)<u(x_i)<$

$u(x_{j+1})$ and $u(x_{j'}) < u(x'_i) < u(x_{j'+1})$ for $j \neq j'$. For the associated measures $m_{[f]}$ and $m_{[f']}$, we have

$$\frac{m_{[f]}}{m_{[f]}(E_i)} = \frac{m_{[f']}}{m_{[f']}(E_i)}.$$

As $m_{[f]}(A_n) = m_{[f']}(A_n) = \nu(A)$, we have $m_{[f]}(E_i) = m_{[f']}(E_i)$, hence $m_{[f]}(E_i) = 1 - m_{[f]}(A_i) = 1 - \nu(A_i \cup A_{j+1} \dots A_n) + \nu(A_{j+1} \dots A_n)$, and $m_{[f']}(E_i) = 1 - m_{[f']}(A_i) = 1 - \nu(A_i \cup A_{j'+1} \dots A_n) + \nu(A_{j'+1} \dots A_n)$ or

$$\nu(A_i \cup A_{j+1} \cup \cdots \cup A_n) - \nu(A_{j+1} \cup \cdots \cup A_n) = \nu(A_i \cup A_{j'+1} \cup \cdots \cup A_n) \\ -\nu(A_{j'+1} \cup \cdots \cup A_n).$$

This is true for any f. Hence, for $i \neq 1, n$, let $A_i = E$, $F = A_{j+1} \cup \cdots \cup A_n$ and $G = A_{j'+1} \cup \cdots \cup A_n$. Clearly, $E \cap F = \varnothing = E \cap G$. The left-hand side of the equality holds if $\nu(A_i \cup A_{j+1} \cup \cdots \cup A_n) - \nu(A_{j+1} \cup \cdots \cup A_n) \neq 1$, i.e. $\nu(F) \neq 0$ and $\nu(F \cup E) \neq 1$ (which ensures that ν_E exists), the right-hand side of the equality holds for every G such that $\nu(G) \neq 0$ and $\nu(G \cup E) \neq 1$ (which ensures that $m(E) \neq 0$). Hence, we get

$$\nu(F \cup E) - \nu(F) = \nu(G \cup E) - \nu(G).$$

(ii) Let us suppose that Property A is satisfied. We partition S in two sets U and U^c (this is possible because of the structure of the null sets) with $\nu(E) \neq 0$ for all non-empty E included in U and $\nu(F) = 0$ for all $F \subset U^c$. Let us note that for all $E \subsetneq U$, $\nu(E) < 1$, the atoms of U are A_i and the ones of N are B_j. We suppose that there are at least three atoms in U.

Let $A_i \subsetneq E \subsetneq U$, $N \subset U^c$. As the complement of $E \cup N$ is not included in U^c, then $0 < \nu(E \cup N) < 1$. We make use of Property A:

$$(a): \nu(E \cup N) - \nu(E \cup N \backslash A_i) = \nu(E) - \nu(E \backslash A_i)$$

From (a) we draw two consequences:

(i) If $\nu(A_1 \cup N) = \nu(A_1) + e$ then for all i, let $E = A_1 \cup A_i$, we get $\nu(A_i \cup N) = \nu(A_i) + e$ (here we use that there are more than three atoms in U);
(ii) For all E, with $0 < \nu(E) < 1$, If $\nu(A_1 \cup N) = \nu(A_1) + e$ then $\nu(E \cup N) = \nu(E) + e$. This can be proved using $A_i \subset E$ and then applying 1) and (a).

Let $\epsilon_j = \nu(A_1 \cup B_j) - \nu(A_1)$. We can distinguish two cases:

- $\epsilon_j = 0$ in which case B_j plays no role; and
- $\epsilon_j \neq 0$ in which case we have $\nu(B_j) = 0$ but $\nu(E \cup B_j) = \nu(E) + \epsilon_j$.

Consider now the capacity ν/U (the restriction of ν to U). It has no other null set than the empty one and fulfills Property A. Therefore we can apply Lemma 3 and conclude that ν/U is a GNAC, so for all $E \subset U$, $\nu/U(E) = a' + b'\pi'(E)$.Let $b = \sum_{i \in I} b'\pi'(A_i) + \sum_{j \in J} \epsilon_j$ and $\pi(A_i) = \frac{b'\pi'(A_i)}{b}$ and $\pi(B_j) = \frac{\epsilon_j}{b}$. We have then:

$\nu(E) = 0$ if $E \subset U^c$,
$\nu(E) = 1$ if $U \subset E$,
$\nu(E) = a + b\pi(E)$ otherwise.

(iii) If ν is a GNAC then Property A is directly satisfied.
(iv) Let us suppose that Property A and is satisfied, that the FBU is used as the updating rule, and show that CCEC is satisfied. Let us consider an act $f_E x$ such that x is not an extreme value of the act $f_E x$. Let us say that $u(x_{i_0}) < u(x) < u(x_{i_0+1})$.

Let $\int f_E x d\nu = \int f_E x dm_{[f_E x]}$, so $m_{[f_E x]}(E) = 1 - m_{[f_E x]}(E^c) = 1 - \nu(E \cup A_{i_0+1} \cup \cdots \cup A_n) + \nu(A_{i_0+1} \cup \cdots \cup A_n)$.

$$\int f d\nu_E = \sum u(x_i)(\nu_E(A_i \cup A_{i+1} \cup \cdots \cup A_n) - \nu_E(A_{i+1} \cup \cdots \cup A_n))$$

As $\nu_E(A) = \frac{\nu(A)}{\nu(A)+1-\nu(E^c \cup A)}$, and by Property A $\nu(E \cup A_{i_0+1} \cup \cdots \cup A_n) - \nu(A_{i_0+1} \cup \cdots \cup A_n) = \nu(E^c \cup A) - \nu(A)$, which implies $\nu(A) + 1 - \nu(E^c \cup A) = m_{[f_E x]}(E)$, we have:

$$\int f d\nu_E = \sum u(x_i) \frac{\nu(A_i \cup A_{i+1} \cup \cdots \cup A_n) - \nu(A_{i+1} \cup \cdots \cup A_n)}{m_{[f_E x]}(E)}$$

Now we must distinguish two cases:

- $u(x) < u(x_i)$, then $m_{[f_E x]}(A_i) = \nu(A_i \cup A_{i+1} \cup \cdots \cup A_n) - \nu(A_{i+1} \cup \cdots \cup A_n)$,
- $u(x) > u(x_i)$ then $m_{[f_E x]}(A_i) = \nu(A_i \cup A_{i+1} \cup \cdots \cup A_n \cup E) - \nu(A_{i+1} \cup \cdots \cup A_n \cup E)$, by Property A, $\nu(A_i \cup A_{i+1} \cup \cdots \cup A_n \cup E) - \nu(A_{i+1} \cup \cdots \cup A_n \cup E) = \nu(A_i \cup A_{i+1} \cup \cdots \cup A_n) - \nu(A_{i+1} \cup \cdots \cup A_n)$

Therefore

$$\int f d\nu_E = \frac{\int_E f dm_{[f_E x]}}{m_{[f_E x]}(E)}$$

$f_E x \sim x \Leftrightarrow \int f_E x d\nu_E = u(x) \Leftrightarrow \int f_E x dm_{[f_E x]} = u(x) \Leftrightarrow \int_E f dm_{[f_E x]} + u(x) m_{[f_E x]}(E^c) = u(x) \Leftrightarrow \int_E f dm_{[f_E x]} = u(x) m_{[f_E x]}(E) \Leftrightarrow \frac{\int_E f dm_{[f_E x]}}{m_{[f_E x]}(E)} = u(x) \Leftrightarrow \int f d\nu_E \Leftrightarrow f \sim_E x$

So CCEC holds □

Proof of Proposition 2: $C(\nu_E)$ is the core of a convex capacity. It is known (see for example Delbaen (1974)) that for any maximal chain (a chain is an ordered set of sets) $C_1 \subset ..C_i.. \subset E$ there exists $\mu \in C(\nu_E)$ such that $\forall i\ \mu(C_i) = \nu_E(C_i)$. $\mu \in P_E$ so for all i there exists $p \in C(\nu)$ such that,

$$\frac{p(C_i)}{p(E)} = \nu_E(C_i) = \frac{\nu(C_i)}{\nu(C_i) + \overline{\nu}(E \backslash C_i)}$$

It thus follows from computations made above that $p(C_i)\overline{\nu}(E \backslash C_i) - \nu(C_i) p(E \backslash C_i) = 0$. As $p(C_i) \geq \nu(C_i)$ and $p(E \backslash C_i) \leq \overline{\nu}(E \backslash C_i)$, we get $p(C_i) = \nu(C_i)$ and $p(E \backslash C_i) = \overline{\nu}(E \backslash C_i)$. From (1) we deduce that for all i,

$$p(E) = \nu(C_i) + \overline{\nu}(E \backslash C_i) = 1 + \nu(C_i) - \nu(C_i \cup E^c)$$

so for A and B non void strictly included in E and ordered by inclusion we have,

$$\nu(E^c \cup A) - \nu(A) = \nu(E^c \cup B) - \nu(B)$$

We can prove it remains true if A and B are not ordered by inclusion. if $A \cap B \neq \emptyset$, we have,

$$\nu(E^c \cup A) - \nu(A) = \nu(E^c \cup (A \cap B)) - \nu(A \cap B) = \nu(E^c \cup B) - \nu(B)$$

if $A \cup B \subset E$ we do the same with $A \cup B$:

$$\nu(E^c \cup A) - \nu(A) = \nu(E^c \cup (A \cup B)) - \nu(A \cup B) = \nu(E^c \cup B) - \nu(B)$$

The remaining case is $A \cup B = E$ and $A \cap B = \emptyset$, if $|E| > 2$, we pick a non void set included in A or B, say A', and get,

$$\nu(E^c \cup A) - \nu(A) = \nu(E^c \cup A') - \nu(A') = \nu(E^c \cup (A' \cup B)) - \nu(A' \cup B)$$
$$= \nu(E^c \cup B) - \nu(B)$$

if $|E| = 2$, as $|S| > 3$ we can write $E^c = F \cup G$ and get,

$$(i): \quad \nu(F \cup G \cup A)) - \nu(G \cup A) = \nu(F \cup G \cup B) - \nu(G \cup B)$$

$$(ii): \quad \nu(G \cup A) - \nu(A) = \nu(G \cup B) - \nu(B)$$

$$(i) - (ii): \quad \nu(E^c \cup A)) - \nu(A) = \nu(E^c \cup B) - \nu(B)$$

So we get the Property A which ensures that ν is a GNAC.

Conversely, let us suppose that ν is a GNAC, we just need to prove that any extreme point μ of $C(\nu_E)$ belongs to P_E. There exits a maximal chain $C_1 \subset ..C_i..C_k \subset E$ such that $\forall i$, $\mu(C_i) = \nu_E(C_i)$. We are going to construct $p \in C(\nu)$ such that for all i,

$$\frac{p(C_i)}{p(E)} = \nu_E(C_i) = \frac{\nu(C_i)}{\nu(C_i) + \overline{\nu}(E \backslash C_i)}$$

On $\mathcal{P}(E)$, the set of parts of E, $\nu_{/E}$, ν restricted to E is a convex capacity, so we can find in its core a probability p such that $p(C_i) = \nu(C_i)$ and $p(E) = \nu(C_k) + \overline{\nu}(E \backslash C_k)$, (compare Delbaen (1974)). By the Hahn–Banach theorem, we can extend p to Σ with p in the core of ν. As ν satisfies Property A we have

$$p(E) = \nu(C_i) + \overline{\nu}(E \backslash C_i) = 1 + \nu(C_i) - \nu(C_i \cup E^c).$$

Hence,

$$p(C_i \cup E^c) = p(E) + p(C_i) = \nu(C_i \cup E^c).$$

Thus, p satisfies Property A and we have $C(\nu_E) = P_E$. □

Proof of Proposition 3: A *chain* is a collection of sets ordered with respect to inclusion. It is maximal when it is maximal for inclusion, i.e., if adding a set to the collection makes it no longer an ordered collection. We make use of the following result (Delbaen 1974, pp. 219–220): C is the core of a convex capacity if and only if for all maximal chains (A_i) there exists $m \in C$ such that $m(A_i) = \min_{p \in C} p(A_i)$. These measures m are the the extreme points of C. Hence, $C = \overline{co}\{m \in C|$ there exists a maximal chain such that $m(A_i) = \min_{p \in C} p(A_i)\}$.

We have to prove $C(\nu_E) \subset P_E$. Let (A_i) be a maximal chain of E and $\mu \in C(\nu_E)$ such that $\mu(A_i) = \min_{p \in C(\nu_E)} p(A_i)$. We want to show that $\mu \in P_E$.

(i) Consider any additive measure m such that $m(A) = \nu(A)$ and $m(E) = 1 - \nu(E \backslash A) + \nu(A)$ and let $p \in C(\nu)$, then

$$\frac{p(A)}{p(E)} - \frac{\nu(A)}{1 - \nu(E^c \cup A) + \nu(A)}$$
$$= \frac{p(A)(1 - \nu(E^c \cup A) + \nu(A)) - \nu(A)(p(E \backslash A) + p(A))}{(1 - \nu(E^c \cup A) + \nu(A))p(E)}$$
$$= \frac{p(A)(1 - \nu(E^c \cup A)) - \nu(A)(p(E \backslash A))}{(1 - \nu(E^c \cup A) + \nu(A))p(E)} \geq 0,$$

since every $p \in C(\nu)$ satisfies $p(A) \geq \nu(A)$ and $p(E \backslash A) \leq 1 - \nu(E^c \cup A)$. Hence,

$$\frac{p(A)}{p(E)} \geq \frac{\nu(A)}{1 - \nu(E^c \cup A) + \nu(A)}$$

(ii) As ν is convex for any chain there exists a measure in its core which is equal to the capacity for each element of the chain. Hence, for the chain A, $A \cup E^c$, S there exists $m \in C(\nu)$ such that $m(A) = \nu(A)$ and $m(E \backslash A) = 1 - \nu(E \backslash A)$. As the set P_E is the core of a convex capacity, for any maximal chain (A_i) of E there exists a measure m such that $m(A_i) = \min_{p \in P_E} p(A_i)$. From (i), m satisfies $m(A_i) = \nu(A_i)$ and $m(E) = 1 - \nu(E \backslash A_i) + \nu(A_i)$ for all i.

Therefore $m = \mu$ and ν is a GNAC according to Proposition 2 □

Proof of Remark 1 Let us check that GNACs satisfy this axiom: let $\arg\min_{s \in S} f(s) \cap \arg\min_{s \in S} g(s) = E_m$ and $\max_{s \in S} f(s) \cap \arg\max_{s \in S} g(s) = E_{M.}$ Let $p = (1 - \delta)\pi + \alpha\delta d_{E_m} + (1 - \alpha)\delta d_{E_M}$, where d_E denotes the Dirac measure of the set E. As $\max\{\min_{s \in S} f(s), \min_{s \in S} g(s)\} \leq \min_{s \in S} h(s)$, $\max_{s \in S^c} h(s) \leq \min\{\max_{s \in S} f(s), \max_{s \in S} g(s)\}$ then $\int f_A h d\nu = \int f_A h dp$ and $\int g_A h d\nu = \int g_A h dp$. As E_m and E_M are included in A, then $\int f d\nu_A = \int f dp_A$ and $\int g d\nu_A = \int g dp_A$. Therefore CCEC comes from that the same measure at every stage. □

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