



# The epsilon-Gini-contamination multiple priors model admits a linear-mean-standard-deviation utility representation

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## Abstract

We introduce the  $\varepsilon$ -Gini-contamination multiple priors model in which the set of priors are those probability measures that are ‘close’ to a central  $P$  with respect to the relative Gini concentration index. We show that such preferences can be represented by a *linear-mean-standard-deviation utility* function on a restricted domain.

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## 1. Introduction and summary

The multiple priors (or Maximin Expected Utility) model, axiomatized by Gilboa and Schmeidler (1989), is a widely applied model that can accommodate situations of general ‘uncertainty’ in which agents perceive there to be some ambiguity or imprecision about which probability law governs the realization of states. A multiple prior preference admits a representation comprising a standard utility

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index  $u$  on outcomes, and (as its name suggests) a set of probabilities  $\mathcal{D}$  on the states on the world and for which the subjective expected utility is calculated with respect to the probability measure that is least favorable for that act. That is, for all pairs of acts  $f$  and  $g$ ,

$$f \succeq g \Leftrightarrow \min_{Q \in \mathcal{D}} \int u(f) dQ \geq \min_{Q \in \mathcal{D}} \int u(g) dQ.$$

One common interpretation given to the set  $\mathcal{D}$  is that it represents the different ‘theories’ the agent considers possible. A further refinement of this interpretation is to suppose that there is a particular probability measure  $P$  that may be viewed as the ‘point’ estimate for the probability law governing the uncertainty. But model uncertainty leads the decision maker to entertain the possibility of other measures. Analogous to how statisticians construct confidence regions,  $\mathcal{D}$  comprises the set of probability measures that are ‘close’ to  $P$  according to some notion of distance. For example, in an  $\varepsilon$ -contamination multiple priors model,  $\mathcal{D}$  is the set of probability measures close to some central  $P$ , in the sense that they are all absolutely continuous with respect to  $P$  and can be expressed as a convex combination of  $P$  and another measure with weights  $1 - \varepsilon$  and  $\varepsilon$ , respectively. Kogan and Tan (2002) take their set  $\mathcal{D}$  to be the set of probability measures whose relative entropy with respect to a given central probability  $P$  is less than a given bound.<sup>1</sup>

In this note, we consider the case where  $\mathcal{D}$  is the set of probability measures that are close to a central  $P$  with respect to the *relative Gini concentration index*. That is, the set  $\mathcal{D}$  is the collection of measures whose relative Gini concentration index with respect to  $P$  is less than some positive bound. We shall refer to this as the  $\varepsilon$ -Gini-contamination multiple priors model.

Our main result (Proposition 2) is that over a suitable ‘domain of monotonicity’ an  $\varepsilon$ -Gini-contamination multiple priors preference relation admits a *linear-mean-standard-deviation-utility (LMSDU)* representation. More precisely, we show

$$\min_{\{Q: G(Q|P) \leq \varepsilon^2\}} \int u(f) dQ = \mu_P(u \circ f) - \varepsilon \sigma_P(u \circ f), \quad (1)$$

where  $G(Q|P)$  is the relative Gini concentration index of  $Q$  with respect to  $P$  (formally, defined in the next section) and  $\mu_P(u \circ f)$  (respectively,  $\sigma_P(u \circ f)$ ) is the mean (respectively, standard deviation) of the random variable  $u \circ f$  with respect to  $P$ .<sup>2</sup> However, the main result also shows that the ‘domain of monotonicity’ restriction is indispensable; that is, for those acts which do not belong to this domain, no function of the form (1) represents the preferences among them.

<sup>1</sup> Although for analytical tractability they restrict all probability measures to be multivariate — normally distributed with the same variance–covariance.

<sup>2</sup> Maccheroni et al. (in press) introduced the relative Gini concentration index as a measure of proximity of one measure to another. Interestingly, they showed that if the ambiguity index in their variational preference model is proportional to this index, then the preferences over a suitably defined sub-domain admit a mean-variance utility representation. That is,

$$\min \left( \int u(f) dQ + \frac{1}{2\theta} G(Q|P) \right) = \mu_P(f) - \frac{\theta}{2} \sigma_P^2(f).$$

They also showed that the relative Gini index is the (Fenchel) dual of a mean - variance function. So the set  $\{Q: G(Q|P) \leq \varepsilon^2\}$  can be written using this duality.

We shall refer to the *RHS* of Eq. (1) as a linear mean-standard-deviation utility (LMSDU) function and any preference relation that admits such as representation as an LMSDU preference relation.

Since  $\mu_p(u \circ f)$  is the expected utility of an act  $f$ , when  $\varepsilon=0$ , LMSDU preferences are simply the standard (Savage-type) preferences. In other words, the class of LMSDU functions is just one parameter richer than the class of subjective expected utility (SEU) functions. One may view an LMSDU function as an augmented SEU function that also explicitly takes account of the standard deviation of utility which has a negative effect on the overall evaluation of an act. The parameter  $\varepsilon$  is the constant marginal rate of substitution between increases in mean utility and reductions in the standard deviation of utility.

Furthermore, an LMSDU function is an instance of a mean-variance utility function widely used in applications in the following sense: let  $X$  be a subset of the real numbers, and let  $u$  be identity. Then by definition  $\mu_P(f)$  is the mean of the random variable  $f$  and  $\sigma_P(f)$  is its standard deviation, and a mean-variance utility function can be written in the form  $\phi(\mu_P(f), \sigma_P(f))$ . So Eq. (1) is a special case where  $\phi$  is linear.<sup>3</sup> Notice, however, that the class of LMSDU preferences differs from mean-variance preferences in the standard sense because the utility index  $u$  may be non-linear. An increase in the risk of *outcomes* not only increases the standard deviation of the induced utility distribution but also affects the expected utility part through the concavity of  $u$ .

Also notice that LMSDU preferences exhibit distribution invariance, that is, to calculate Eq. (1), one only needs to know the distribution of utility induced by the random variable  $u \circ f$  and the underlying probability measure  $P$ .<sup>4</sup> Therefore, our main result (Proposition 2) shows that an LMSDU function may be regarded as a distribution invariant representation of a multiple priors preference relation. For the special case where  $\Omega$  is finite,  $u$  is linear, and  $P$  is the uniform measure, Quiggin and Chambers (1998) pointed out that an LMSDU function is a representation of a  $\varepsilon$ -Gini-contamination multiple priors preference relation. In contrast, we concentrate on the direct link between an  $\varepsilon$ -Gini-contamination multiple priors preference relation and an LMSDU function, without imposing any structure on  $\Omega$ ,  $u$  or  $P$ .

It will be seen that when  $\Omega$  is finite and  $\varepsilon$  is small enough, the domain of monotonicity contains all acts, but it is strictly smaller than the set of all acts if the state space is rich enough. So a more important contribution of this note is to point out that such a representation result cannot extend to the domain of all acts when the state space is infinite. That is, although both an  $\varepsilon$ -Gini-contamination multiple priors preference relation and an  $\varepsilon$ -LMSEU can be defined for a general state space, they are related only on a restricted domain.

In the next section we set up the formal framework and then state and prove our main result.

## 2. Multiple priors and the relative Gini concentration index

Let  $\Omega$  be a set of states (finite or infinite) and let  $\Sigma$  be a  $\sigma$ -field on  $\Omega$ . Let  $X$  be a set of outcomes. An act is a  $\Sigma$ -measurable function with finite range; that is, an act is a function  $f: \Omega \rightarrow X$  with  $\{f^{-1}(x) : x \in X\}$  constituting a partition of  $\Omega$  with finitely many  $\Sigma$ -measurable sets. The set of all acts is denoted by  $\mathcal{F}$ .

<sup>3</sup> Quiggin and Chambers (2004) identified the condition where  $\phi$  is linear.

<sup>4</sup> If  $P$  were also convex-ranged, that is, for all  $P(E) > 0$  and  $\alpha$  in  $(0, 1)$ , there exists an event  $E' \subseteq E$  such that  $P(E') = \alpha P(E)$ , then the LMSDU preferences would be *probabilistically sophisticated* in the sense of Machina and Schmeidler (1992).

Denote by  $\mathcal{P}$  the set of all probability measures. For given  $Q \in \mathcal{P}$  and utility index  $u: X \rightarrow \mathbb{R}$ , and an act  $f \in \mathcal{F}$ , let  $\mu_Q(u \circ f)$  denote the mean of  $u \circ f$  with respect to the measure  $Q$ :

$$\mu_Q(u \circ f) = \int u(f) dQ = \sum_{x \in X} u(x) Q(f^{-1}(x)),$$

Similarly, let  $\sigma_Q(u \circ f)$  denote the standard deviation of  $u \circ f$  with respect to  $Q$ :

$$\sigma_Q(u \circ f) = \left[ \sum_{x \in X} [u(x) - \mu_Q(u \circ f)]^2 Q(f^{-1}(x)) \right]^{1/2}.$$

Note that both  $\mu_Q(u \circ f)$  and  $\sigma_Q(u \circ f)$  are well defined for any  $Q \in \mathcal{P}$ , any utility index  $u: X \rightarrow \mathbb{R}$ , and any act  $f \in \mathcal{F}$ , since every act is finite ranged.

Fix  $P, Q$ , in  $\mathcal{P}$ . Define the relative Gini Concentration index as follows:  $G(Q|P) = \infty$  if  $Q$  is not absolutely continuous with respect to  $P$ , otherwise, writing  $dQ/dP$  for the Radon Nykodin derivative,

$$G(Q|P) = \int \left( \left( \frac{dQ}{dP} \right) - 1 \right)^2 dP = \int \left( \frac{dQ}{dP} \right) dQ - 1. \quad (2)$$

It can be readily seen that  $G(Q|P) \geq 0$  and  $G(Q|P) = 0$  if and only if  $P = Q$ , hence it is also known as the  $\chi^2$ -distance  $Q$  is from  $P$ .

If  $\Omega$  is finite, then Eq. (2) is reduced to  $\sum_{\omega} \left( \frac{Q(\omega)}{P(\omega)} \right) Q(\omega) - 1$ . Thus for any finite partition of  $\Omega$ ,  $[E_1, \dots, E_n]$ , it readily follows

$$\sum_{j=1}^n \frac{Q(E_j)}{P(E_j)} Q(E_j) - 1 \leq G(Q|P).$$

Fix  $\varepsilon > 0$ , and consider the set  $\mathcal{G}(P, \varepsilon)$ , defined as

$$\mathcal{G}(P, \varepsilon) = \{Q \in \mathcal{P} : G(Q|P) \leq \varepsilon^2\}.$$

Notice that this is the set of priors ‘centered’ around  $P$  for which the relative Gini concentration index (or  $\chi^2$ -distance) is not greater than  $\varepsilon^2$ .

**Example 1.** Fix  $\Omega = \{\omega_1, \dots, \omega_n\}$ . take  $P$  to be uniform, that is,  $P(\omega_i) = 1/n$ ,  $i = 1, \dots, n$ . Then

$$\mathcal{G}(P, \varepsilon) = \left\{ Q \in \mathcal{P} : \sum_{i=1}^n (Q(\omega_i) - 1/n)^2 \leq \varepsilon^2/n \right\}.$$

Geometrically,  $\mathcal{G}(P, \varepsilon)$  resides in the hyper-plane  $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1\}$  and is the intersection of the unit simplex with a disc of radius  $\varepsilon/n$  centered at  $(1/n, \dots, 1/n)$ .

Let us consider the multiple prior preference  $\succsim$  generated by the function

$$V(f) = \inf_{Q \in \mathcal{G}(P, \varepsilon)} \int u(f) dQ. \tag{3}$$

Since the Gini index is the average distance of two measure, it does not imply any uniform bound on the measures. In particular, if a signed measure  $Q$  with  $Q(\Omega) = 1$  is close to a probability measure  $P$  in terms of its relative Gini concentration index  $G(Q|P)$ , then  $Q$  may not be positive, that is, it need not to be a probability measure. Indeed if  $P$  is convex-ranged (that is, has no atoms) then for any  $\varepsilon > 0$ , there exists a signed measure  $Q \notin \mathcal{P}$  with  $Q(\Omega) = 1$  for which  $G(Q|P) \leq \varepsilon^2$ . Thus in general, to solve the minimization problem in Eq. (3), it is not enough to restrict to measures for which  $G(Q|P) \leq \varepsilon^2$ , we also need to check non-negativity constraints as well.<sup>5</sup> It is only for sufficiently coarse partitions of  $\Omega$ , that all measures that are within an  $\varepsilon^2$   $\chi^2$ -distance of  $P$ , are automatically positive. Formally, let  $\Pi$  be a finite partition of  $\Omega$  in  $\Sigma$ -measurable sets. We say that  $\Pi$  is  $\varepsilon$ -Gini-coarse with respect to  $P$  if  $\min \{P(E) : E \in \Pi, P(E) > 0\} > \varepsilon^2 / (1 + \varepsilon^2)$ .

**Lemma 1.** *If a partition  $\Pi$  is  $\varepsilon$ -Gini-coarse with respect to  $P$ , then for any  $Q \in \mathcal{P}$*

$$\sum_{E \in \Pi, P(E) > 0} \left( \frac{Q(E)}{P(E)} \right) Q(E) - 1 \leq \varepsilon^2 \text{ implies } Q(E) > 0 \text{ for all } E \in \Pi, \text{ s.t. } P(E) > 0.$$

**Proof.** We first recall the following simple fact: let  $a_i > 0$ ,  $i = 1, \dots, n$  and set  $\bar{a} = \sum_{j=1}^n a_j$ . Consider minimizing the following function

$$g(x) = \left( \sum_i \frac{x_i^2}{a_i} \right) - 1$$

on the set  $X := \{x \in \mathbb{R}^n : \sum x_i = 1\}$ . Since  $g$  is convex, this problem can be solved by examining the Kuhn–Tucker condition, which yields  $x_i/a_i = 1/\bar{a}$  for every  $i$ , and consequently we have

$$\min g(x) = \left( \sum_i \left( \frac{1}{\bar{a}} \right)^2 a_i \right) - 1 = \frac{1}{\bar{a}} - 1.$$

<sup>5</sup> If there is further restrictions on the set of probabilities, such as they are normally distributed as in Kogan and Tan (2002), closeness in the relative Gini index can ensure positivity.

Suppose that there exists a  $Q$  with  $\sum_{E \in \Pi, P(E) > 0} \left(\frac{Q(E)}{P(E)}\right) Q(E) - 1 < \varepsilon$  such that  $Q(A) = 0$  for some  $A \in \Pi$  with  $P(A) > 0$ . Then  $Q$  is included in the set  $\{Q' \in \mathcal{P} : Q'(A) = 0\}$ , and so in particular  $P(A) = 1$  is impossible. Consider a probability measure which minimizes the expression

$$\sum_{E \in \Pi, P(E) > 0} \left(\frac{Q(E)}{P(E)}\right) Q(E) - 1$$

on the set  $\{Q' \in \mathcal{P} : Q'(A) = 0\}$ . From the observation above, the minimum is attained by measure  $\hat{Q}$  with the property

$$\left(\frac{\hat{Q}(E)}{P(E)}\right) = \frac{1}{\sum_{E' \in \Pi, E' \neq A} P(E')} = \frac{1}{1 - P(A)} \text{ for all } E \in \Pi, E \neq A, P(E) > 0,$$

and

$$\min_{Q' \in \mathcal{P}, Q'(A) = 0} \left( \sum_{E \in \Pi, P(E) > 0} \left(\frac{\hat{Q}(E)}{P(E)}\right) \hat{Q}(E) - 1 \right) = \frac{P(A)}{1 - P(A)} > \varepsilon^2,$$

where last inequality follows since  $P(A) > \varepsilon^2 / (1 + \varepsilon^2)$  and hence  $(1 - P(A))^{-1} > (1 + \varepsilon^2)$ . But this is a contradiction since  $\sum_{E \in \Pi, P(E) > 0} \left(\frac{Q(E)}{P(E)}\right) Q(E) - 1 \leq \varepsilon^2$  and  $Q(A) = 0$ .  $\square$

Notice that if a partition  $\Pi$  is  $\varepsilon$ -Gini-coarse with respect to  $P$  then partition that is a coarsening of  $\Pi$  must also be  $\varepsilon$ -Gini-coarse with respect to  $P$ .

Say that an act  $f$  is  $\varepsilon$ -Gini-coarse with respect to  $P$  if the partition  $\{f^{-1}(x) : x \in X\}$  is  $\varepsilon$ -Gini-coarse with respect to  $P$ . Let  $\mathcal{F}(P, \varepsilon)$  be the set of acts that are  $\varepsilon$ -Gini-coarse with respect to  $P$ , or equivalently,

$$\mathcal{F}(P, \varepsilon) = \{f \in \mathcal{F} : \text{for all } x \in X, P(f^{-1}(x)) = 0 \text{ or } P(f^{-1}(x)) > \varepsilon^2 / (1 + \varepsilon^2)\}.$$

That is, for every outcome in the range of  $f$ , either its inverse image is a null event with respect to  $P$  or an event that  $P$  assigns a probability greater than  $\varepsilon^2 / (1 + \varepsilon^2)$ . Notice that for any  $\varepsilon > \varepsilon' \mathcal{F}(P, \varepsilon) \subset \mathcal{F}(P, \varepsilon')$ . And that the set of all acts  $\mathcal{F}$  is simply  $\mathcal{F}(P, 0)$ .

Furthermore, if  $\Omega$  is finite and  $\min \{P(\omega) : \omega \in \Omega, P(\omega) > 0\} > \varepsilon^2 / (1 + \varepsilon^2)$  then all acts are in  $\mathcal{F}(P, \varepsilon)$ , that is,  $\mathcal{F} = \mathcal{F}(P, \varepsilon)$ . As an illustration, recall in example 1 above,  $P$  is a uniform distribution on a finite state space of cardinality  $n$ . Hence, provided  $\varepsilon^2 / (1 + \varepsilon^2) < 1/n$  (that is  $\varepsilon < 1/\sqrt{n-1}$ ), the disc of radius  $\varepsilon/\sqrt{n}$  centered at  $(1/n, \dots, 1/n)$  lies inside the unit simplex. In this case  $\mathcal{G}(P, \varepsilon)$  coincides with this disc. Hence, for each  $Q$  in  $\mathcal{F}(P, \varepsilon)$ ,  $Q(\omega) > 0$ , for every  $\omega$  in  $\Omega$  and thus,  $\mathcal{F}(P, \varepsilon) = \mathcal{F}$ .

We are now ready to state the main result.

**Proposition 2.** Fix  $P$  and  $\varepsilon$ . Let  $\succsim$  be the multiple priors preference relation generated by the function

$$V(f) = \inf_{Q \in \mathcal{G}(P, \varepsilon)} \int \mu(f) dQ.$$

The restriction of  $\succsim$  to  $\mathcal{F}(P, \varepsilon)$  admits the representation

$$V(f) = \mu_P(\mu \circ f) - \varepsilon \sigma_P(u \circ f).$$

Furthermore, (i) for any act  $f$

$$V(f) \geq \mu_P(\mu \circ f) - \varepsilon \sigma_P(u \circ f),$$

and (ii) for any  $\Pi$  that is not  $\varepsilon$ -Gini-coarse with respect to  $P$ , there exist an act  $g$  that is measurable with respect to  $\Pi$  for which

$$V(g) > \mu_P(u \circ g) - \varepsilon \sigma_P(u \circ g).$$

**Proof.** Fix  $f \in \mathcal{F}(P, \varepsilon)$ . The minimization problem can be reexpressed as follows:

$$V(f) = \inf_{Q \in \mathcal{G}(P, \varepsilon)} \int \mu(f) dQ = \mu_P(u \circ f) - \sup_{Q \in \mathcal{G}(P, \varepsilon)} \sum_{x \in X} [\mu_Q(u \circ f) - u(x)] Q(f^{-1}(x)).$$

For the maximization problem on the second line above, note that by Lemma 1, we can ignore the non-negativity constraints, that is  $Q(f^{-1}(x)) \geq 0$  for all  $x$ . So in effect, the relevant Lagrangian can be set up as follows: assign  $\lambda/2$  as the Lagrange multiplier to the constraint

$$\sum_{\{x: P(f^{-1}(x)) > 0\}} \left( \frac{Q(f^{-1}(x))}{P(f^{-1}(x))} \right) Q(f^{-1}(x)) - 1 \leq \varepsilon^2. \tag{4}$$

and  $\gamma$  as the Lagrange multiplier to the constraint

$$\sum_{\{x: P(f^{-1}(x)) > 0\}} Q(f^{-1}(x)) = 1, \tag{5}$$

and then we have:

$$\begin{aligned} \mathcal{L} = & \sum_{\{x: P(f^{-1}(x)) > 0\}} [\mu_P(u \circ f) - u(x)] Q(f^{-1}(x)) \\ & - \frac{\lambda}{2} \left[ \sum_{\{x: P(f^{-1}(x)) > 0\}} \left( \frac{Q(f^{-1}(x))}{P(f^{-1}(x))} \right) Q(f^{-1}(x)) - 1 - \varepsilon^2 \right] - \gamma \left( 1 - \sum_{\{x: P(f^{-1}(x)) > 0\}} Q(f^{-1}(x)) \right). \end{aligned}$$

This is a standard (finite) problem with a concave objective function with variables  $\{Q(f^{-1}(x))\}$  on convex sets, so the first order conditions fully characterize the solution, which are:

$$Q(f^{-1}(x)) : \mu_P(u \circ f) - u(x) = \lambda \frac{Q(f^{-1}(x))}{P(f^{-1}(x))} - \mu, \tag{6}$$

for every  $x$  such that  $P(f^{-1}(x)) > 0$ . Multiplying Eq. (6) by  $P(f^{-1}(x))$  and summing over  $x$  such that  $P(f^{-1}(x)) > 0$  we obtain:

$$\sum_{\{x:P(f^{-1}(x))>0\}} [\mu_P(u \circ f) - u(x)]P(f^{-1}(x)) = \sum_{\{x:P(f^{-1}(x))>0\}} (\lambda Q(f^{-1}(x)) - \gamma P(f^{-1}(x))),$$

Hence from constraint (5) it follows

$$\lambda = \gamma. \quad (7)$$

By squaring Eq. (6), then multiplying by  $P(f^{-1}(x))$  and summing over  $x$  such that  $P(f^{-1}(x)) > 0$ , we obtain:

$$\begin{aligned} & \sum_{\{x:P(f^{-1}(x))>0\}} [u(x) - \mu_P(u \circ f)]^2 P(f^{-1}(x)) \\ &= \lambda^2 \times \sum_{\{x:P(f^{-1}(x))>0\}} \left( \frac{Q(f^{-1}(x)) - P(f^{-1}(x))}{P(f^{-1}(x))} \right)^2 P(f^{-1}(x)). \end{aligned} \quad (8)$$

The LHS of Eq. (8) is simply  $\sigma_P^2(u \circ f)$ . Expanding the RHS, this yields

$$\begin{aligned} & \lambda^2 \times \sum_{x \in \{y \in X: P(f^{-1}(y)) > 0\}} \frac{Q(f^{-1}(x))^2 - 2Q(f^{-1}(x))P(f^{-1}(x)) + P(f^{-1}(x))^2}{P(f^{-1}(x))} \\ &= \lambda^2 \left[ \sum_{x \in \{y \in X: P(f^{-1}(y)) > 0\}} \left( \frac{Q(f^{-1}(x))}{P(f^{-1}(x))} \right) Q(f^{-1}(x)) - 2 + 1 \right] = \lambda^2 \varepsilon^3 \text{ (from (4)).} \end{aligned}$$

Hence,

$$\lambda = \sigma_P(u \circ f) / \varepsilon. \quad (9)$$

Substituting Eqs. (7) and (9) into Eq. (6) yields,

$$Q(f^{-1}(x)) = P(f^{-1}(x)) \left[ 1 - \frac{\varepsilon}{\sigma_P(u \circ f)} [u(x) - \mu_P(u \circ f)] \right]. \quad (10)$$

So, we have:

$$\begin{aligned} & \max_{Q \in \mathcal{G}(\pi, \varepsilon)} \sum_{\{x:P(f^{-1}(x))>0\}} [\mu_P(u \circ f) - u(x)]Q(f^{-1}(x)) \\ &= \sum_{\{x:P(f^{-1}(x))>0\}} [\mu_P(u \circ f) - u(x)]P(f^{-1}(x)) \\ & \quad + \varepsilon \times \sum_{\{x:P(f^{-1}(x))>0\}} [\mu_P(u \circ f) - u(x)]^2 P(f^{-1}(x)) / \sigma_P(u \circ f) \\ &= \frac{\varepsilon}{\sigma_P(u \circ f)} \sum [EU_P(f) - u(x)]^2 P(f^{-1}(x)) = \frac{\varepsilon}{\sigma_P(u \circ f)} \times \sigma_P^2(u \circ f) = \varepsilon \times \sigma_P(u \circ f). \end{aligned}$$

Thus,

$$V(f) = \mu_P(u \circ f) - \varepsilon \sigma_P(u \circ f).$$

as we wanted.

To establish the second half, note first that if  $f \notin \mathcal{F}(P, \varepsilon)$ , the non-negativity constraint we ignored above may well be binding. But the introduction of the non-negativity constraint makes infimum no smaller.

Now suppose there exists a partition  $\Pi$  that is not  $\varepsilon$ -Gini-coarse with respect to  $P$ . That is, there exists an event  $A \in \Pi$ , for which  $0 < P(A) < \varepsilon^2 / (1 + \varepsilon^2)$ , and hence  $[1 - P(A)]^{-1} < (1 + \varepsilon^2)$ .

Consider the act  $g$ , where  $g(\omega) = x$  if  $\omega \in A$  and  $g(\omega) = y$ , if  $\omega \notin A$ , and where  $u(x) > u(y)$ . By construction it is measurable with respect to  $\Pi$ , and by state-monotonicity,

$$\begin{aligned} V(g) &\geq u(y) > u(y) + \left( P(A) - \varepsilon \sqrt{P(A)(1-P(A))} \right) (u(x) - u(y)) \left( \text{since } \frac{P(A)}{1-P(A)} < \varepsilon^2 \right) \\ &= P(A)u(x) + [1-P(A)]u(y) - \varepsilon \sqrt{P(A)(1-P(A))} (u(x) - u(y)) = \mu_P(u \circ g) - \varepsilon \sigma_P(u \circ g). \quad \square \end{aligned}$$

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