Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 130 (2006) 264–282
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Notes, Comments, and Letters to the Editor

Bayesian beliefs with stochastic monotonicity: An extension of Machina and Schmeidler

Simon Grant[a,b], Ben Polak[c,*]

[a]*Department of Economics, Baker Hall, Rice University, MS 22, PO Box 1892, Houston, TX 77251-1892, USA*
[b]*School of Economics, Faculty of Economics & Commerce, Australian National University, ACT 0200, Australia*
[c]*Department of Economics, Yale University, USA*

Received 19 May 2004; final version received 17 March 2005
Available online 31 May 2005

Abstract

Machina and Schmeidler show that the probabilistic sophistication can be obtained in an Anscombe–Aumann setting without imposing expected utility by maintaining stochastic monotonicity and adding a new axiom loosely analogous to Savage's P4. This analogous axiom, however, is very strong. In this note, we obtain probabilistic sophistication using a weaker (and more natural) analog of Savage's P4. Stochastic monotonicity is sufficient to bridge the gap, where Anscombe and Aumman use independence twice, we use stochastic monotonicity twice.

JEL classification: D81

Keywords: Subjective probability; Stochastic monotonicity; Independence; Horse/roulette lotteries; Anscombe–Aumann

1. Introduction

In Subjective Expected Utility Theory, an agent's preferences under uncertainty can be represented by the expectation of a utility function with respect to the subjective probabilities that represent the individual's beliefs. Savage [9] axiomatized this in a setting of purely

* Corresponding author.
E-mail addresses: sgrant@rice.edu (S. Grant), benjamin.polak@yale.edu (B. Polak).

0022-0531/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jet.2005.03.006

subjective uncertainty while Anscombe and Aumann [1] adopted a setting of mixed subjective and objective uncertainty that afforded them a simpler treatment. Anscombe and Aumann argued that the novelty of their presentation lay in the double use of the Independence Axiom: first to scale risk preferences and so obtain the agent's von Neumann–Morgenstern utility index, and second to scale events and so obtain the agent's subjective probabilities. Machina and Schmeidler's [7,8] contribution was to provide axiomatizations that continued to characterize beliefs by means of subjective probabilities, but neither assumed nor implied that risk preferences conformed to Expected Utility Theory. In [7], they did this in the subjective uncertainty setting of Savage while dropping the Sure-Thing Principle. In [8], they adapt the mixed subjective and objective uncertainty of Anscombe and Aumann while dropping Independence. [1]

Since Machina and Schmeidler [8] work in the Anscombe–Aumann framework, they can use objective probabilities to calibrate the likelihood of events. They introduce two new axioms. One ensures that the agent's preferences are Stochastically Monotonic; that is, they respect First-order Stochastic Dominance. The second ensures that the probability assessments are invariant to the outcomes used in the calibrations. This second axiom is loosely analogous to Savage's axiom P4. Savage, however, had available the Sure-Thing Principle, and Anscombe and Aumann obtained the Sure-Thing Principle from Independence. In the absence of either Independence or the Sure-Thing Principle, Machina and Schmeidler strengthen their calibration axiom (their analogy to Savage's P4) to incorporate properties that would have followed from Independence. The purpose of this note is to show that we can work with a weaker (and more natural) analog of Savage's P4. We show that the gap can be filled using just Stochastic Monotonicity. That is, Anscombe and Aumann used Independence twice; first to scale risk preferences, and second to calibrate subjective probabilities. We use Stochastic Monotonicity twice; first again to scale risk preferences, and second to extend our analog of Savage's P4 and hence to calibrate subjective probabilities.

In Section 2, we first outline Fishburn's rendering of the Anscombe–Aumann setting. We then present Machina and Schmeidler's set of conditions sufficient to imply Probabilistic Sophistication. Section 3 contains and discusses our alternative conditions, and presents the main theorem. Section 4 provides examples to demonstrate the independence of axioms.

2. Preliminaries

2.1. Setup

Assume that uncertainty is described by a set of states, $\mathcal{S}$. This set may be finite or infinite. For any event $E \subset \mathcal{S}$, let E^{c} denote its complement. Let $\mathcal{X}$ be an arbitrary set of outcomes (finite or infinite) and denote by $\mathcal{L}$ the set of probability measures on $\mathcal{X}$ with finite supports. We will refer to the elements of $\mathcal{L}$ as roulette lotteries or just as lotteries. For each x in $\mathcal{X}$,

[1] Recently, Machina [6] shows that an Anscombe–Aumann-like analysis can be conducted without any objective uncertainty by replacing the roulette lotteries with acts that are measurable with respect to "almost-objective" events. Among the many results in that paper, he adapts the axioms and analysis of Machina and Schmeidler [8] to that framework again deriving probabilitistic sophistication without expect utility. The axioms and analysis in this note could be similarly adapted to apply in that setting.

let δ_x denote the (degenerate) lottery that yields the outcome x with probability one. Any lottery $\mathbf{R}$ in $\mathcal{L}$ may be expressed as a probability weighted mixture of degenerate lotteries, corresponding to the outcomes in its support: that is, $\sum_{i=1}^{m} p_i \delta_{x_i}$, where $p_i = \mathbf{R}(x_i)$ for each $i = 1, \ldots, m$.

A horse/roulette lottery is a function $\mathbf{H} : \mathcal{S} \to \mathcal{L}$ with finite range. Thus $\mathbf{H}(s)(x)$ is the probability assigned to the outcome x by the roulette lottery that obtains in state s let $\mathcal{H}$ be the set of horse/roulette lotteries. It is often convenient to associate with a horse/roulette lottery $\mathbf{H} \in \mathcal{H}$, a (finite) partition $\{E_1, \ldots, E_n\}$ of $\mathcal{S}$ on which $\mathbf{H}$ is measurable. Thus $\mathbf{H}$ may be expressed as $\mathbf{H} := [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$. Furthermore, for any pair of acts $\mathbf{H}^*$ and $\mathbf{H}^{**}$ in $\mathcal{H}$ and any event $E \subset \mathcal{S}$, $[\mathbf{H}^* \text{ on } E; \mathbf{H}^{**} \text{ on } E^c]$ will denote the horse/roulette lottery $\mathbf{H}$ in $\mathcal{H}$ formed from 'splicing' and 'recombining' the two acts $\mathbf{H}^*$ and $\mathbf{H}^{**}$ in such a way that $\mathbf{H}(s)$ equals $\mathbf{H}^*(s)$ if $s \in E$, and equals $\mathbf{H}^{**}(s)$ if $s \notin E$.

Both the sets $\mathcal{L}$ and $\mathcal{H}$ are convex. In particular, for any pair of roulette lotteries $\mathbf{R}^*$ and $\mathbf{R}^{**}$ in $\mathcal{L}$, and any α in (0, 1), $\alpha\mathbf{R}^* + (1-\alpha)\mathbf{R}^{**}$ is the roulette lottery $\mathbf{R}$ in $\mathcal{L}$ for which $\mathbf{R}(x) = \alpha\mathbf{R}^*(x) + (1-\alpha)\mathbf{R}^{**}(x)$ for each x in $\mathcal{X}$. Similarly, for any pair of horse/roulette lotteries $\mathbf{H}^*$ and $\mathbf{H}^{**}$ in $\mathcal{H}$, and any α in (0, 1), take $\alpha\mathbf{H}^* + (1-\alpha)\mathbf{H}^{**}$ to be the horse/roulette lottery $\mathbf{H} \in \mathcal{H}$, in which $\mathbf{H}(s) = \alpha\mathbf{H}^*(s) + (1-\alpha)\mathbf{H}^{**}(s)$, for each s in $\mathcal{S}$.

Let $\succ$ denote the individual's strict preference relation on $\mathcal{H}$. Following Fishburn, we shall take strict preference as the basic binary relation. Indifference, denoted by $\sim$, is then defined as the absence of strict preference, and weak preference, denoted by $\succsim$, is defined as the union of strict preference and indifference. An event E is deemed null (with respect to $\succ$) if $\left[\mathbf{H} \text{ on } E;\ \mathbf{H}^* \text{ on } E^c\right] \sim \mathbf{H}^*$ for all $\mathbf{H}, \mathbf{H}^* \in \mathcal{F}$. Let $\mathcal{N}$ denote the set of null events.

Let $\geqslant^1$ (respectively, $>^1$) denote the partial ordering over roulette lotteries of (respectively, strict) First-order Stochastic Dominance derived from $\succ$. It is defined as follows: for any pair of lotteries $\mathbf{R} = \sum_{i=1}^{m} p_i \delta_{x_i}$ and $\mathbf{R}^* = \sum_{j=1}^{m^*} q_j \delta_{y_j}$, $\mathbf{R} \geqslant^1 \mathbf{R}^*$, if

$$\sum_{\{i:\delta_{x_i} \succ \delta_z\}} p_i \geqslant \sum_{\left\{j:\delta_{y_j} \succ \delta_z\right\}} q_j \quad \text{for all } z \in \mathcal{X}.$$

And $\mathbf{R} >^1 \mathbf{R}^*$, if, in addition, strict inequality holds for some $z \in \mathcal{X}$.

With slight abuse of notation, x will also denote the degenerate roulette lottery δ_x. Similarly, $\mathbf{R}$ will also denote the constant horse/roulette lottery $[\mathbf{R} \text{ on } \mathcal{S}]$; that is, the horse/roulette lottery $\mathbf{H}$ such that $\mathbf{H}(s) = \mathbf{R}$ for all s in $\mathcal{S}$. Thus, x also denotes the constant horse/roulette lottery $\left[\delta_x \text{ on } \mathcal{S}\right]$. Hence, we will write $x \succ y$ if and only if $\left[\delta_x \text{ on } \mathcal{S}\right] \succ \left[\delta_y \text{ on } \mathcal{S}\right]$, and $\mathbf{R} \succ \mathbf{R}^*$ if and only if $[\mathbf{R} \text{ on } \mathcal{S}] \succ \left[\mathbf{R}^* \text{ on } \mathcal{S}\right]$. We shall refer to the restriction of $\succ$ to constant horse/roulette lotteries as the individual's risk preferences. We shall say the risk preferences satisfy Stochastic Monotonicity if they respect the partial ordering of First-order Stochastic Dominance, that is, $\mathbf{R} >^1 \mathbf{R}^*$ implies $\mathbf{R} \succ \mathbf{R}^*$ and $\mathbf{R} \geqslant^1 \mathbf{R}^*$ implies $\mathbf{R} \succsim \mathbf{R}^*$. We say that a function $V : \mathcal{L} \to \mathbb{R}$ is Mixture Continuous and Stochastically Monotonic if it is continuous in probability mixtures and $\mathbf{R} >^1 \left(\geqslant^1\right) \mathbf{R}^*$ implies $V(\mathbf{R}) > (\geqslant)\ V(\mathbf{R}^*)$. Furthermore, we say it is affine if for any pair of roulette lotteries $\mathbf{R}$ and $\mathbf{R}^*$ in $\mathcal{L}$, and any α in (0, 1), $V(\alpha\mathbf{R} + (1-\alpha)\mathbf{R}^*) = \alpha V(\mathbf{R}) + (1-\alpha) V(\mathbf{R}^*)$.

2.2. Subjective expected utility

A preference relation $\succ$ on $\mathcal{H}$ is said to admit a Subjective Expected Utility representation, if there exists a unique subjective probability measure $\pi(.)$ over events and an affine utility function $U(.)$ on $\mathcal{L}$, unique up to positive affine transformations, such that, for all pairs of horse/roulette lotteries and $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$ and $\mathbf{H}^* = [\mathbf{R}_1^* \text{ on } E_1^*; \ldots; \mathbf{R}_{n^*}^*$ on $E_{n^*}^*]$

$$\mathbf{H} \succ \mathbf{H}^* \Leftrightarrow U\left(\sum_{i=1}^{n} \pi(E_i)\, \mathbf{R}_i\right) > U\left(\sum_{j=1}^{n^*} \pi\left(E_j^*\right) \mathbf{R}_j^*\right).$$

We follow Fishburn's [2, Theorem 3.3, p. 179] rendering of the Anscombe–Aumann theorem. The following set of axioms is necessary and sufficient for a preference relation to admit a Subjective Expected Utility representation. [2]

Ordering: *The relation $\succ$ on $\mathcal{H}$ is a weak order.*

Archimedean: *For any three horse/roulette lotteries* $\mathbf{H}, \mathbf{H}^*$ *and* $\mathbf{H}^{**}$ *in* $\mathcal{H}$, $\mathbf{H}^* \succ \mathbf{H} \succ \mathbf{H}^{**}$ *implies*

$$\alpha \mathbf{H}^* + (1-\alpha)\, \mathbf{H}^{**} \succ \mathbf{H} \succ \beta \mathbf{H}^* + (1-\beta)\, \mathbf{H}^{**}$$

for some $\alpha, \beta \in (0, 1)$.

Non-Degeneracy: *There exist outcomes* x *and* y *in* $\mathcal{X}$, *such that* $x \succ y$.

Independence: *For any three horse/roulette lotteries* $\mathbf{H}, \mathbf{H}^*$ *and* $\mathbf{H}^{**}$ *in* $\mathcal{H}$, *and* α *in* $(0, 1]$,

$$\mathbf{H} \succ \mathbf{H}^* \Leftrightarrow \alpha \mathbf{H} + (1-\alpha)\, \mathbf{H}^{**} \succ \alpha \mathbf{H}^* + (1-\alpha)\, \mathbf{H}^{**}.$$

Substitution: *For any pair of roulette lotteries* $\mathbf{R}$ *and* $\mathbf{R}^*$ *in* $\mathcal{L}$, *any horse/roulette lottery* $\mathbf{H}$ *in* $\mathcal{H}$ *and any non-null event* $E \notin \mathcal{N}$,

$$\mathbf{R} \succ \mathbf{R}^* \Leftrightarrow [\mathbf{R}\, on\, E;\ \mathbf{H}\, on\, \sim E] \succ \left[\mathbf{R}^*\, on\, E;\ \mathbf{H}\, on\, \sim E\right].$$

The first two axioms are analogous to the standard ordering and continuity axioms in choice under certainty. The third axiom simply requires the individual not be indifferent between all outcomes. As Machina and Schmeidler observe, the next two axioms (Independence and Substitution) may be viewed as being "expected utility-based". That is, not only

[2] Fishburn also has a state-wise dominance axiom (B6) but it is not needed in this setting in which every roulette lottery has finite support and every horse/roulette lottery has finite range.

do they ensure that beliefs can be represented by a unique subjective probability measure but they also entail that risk preferences are represented by a von Neumann–Morgenstern affine utility function. [3] The aim of Machina and Schmeidler [8] was to investigate whether it was possible to have the former and without necessarily entailing the latter.

We now formally define probabilistic sophistication. Suppose we fix a probability measure $\hat{\pi}$ over events, then we can associate with any act $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$, the constant horse/roulette lottery $\sum_{i=1}^{n} \hat{\pi}(E_i)\,\mathbf{R}_i$, that is, the roulette lottery formed by taking the $\hat{\pi}-$weighted average of the lotteries in the range of $\mathbf{H}$. Probabilistic sophistication means that there exists a unique subjective probability measure π defined over the set of events, such that for any pair of acts $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$ and $\mathbf{H}^* = [\mathbf{R}_1^* \text{ on } E_1^*; \ldots; \mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*]$, $\mathbf{H}$ is strictly preferred to $\mathbf{H}^*$ if and only if the constant horse/roulette lottery $\sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i$ is strictly preferred to the constant horse/roulette lottery $\sum_{j=1}^{n^*} \pi\left(E_j^*\right)\mathbf{R}_j^*$. That is, if an agent is probabilistically sophisticated then knowledge of an individual's subjective beliefs π and her risk preferences (that is, the restriction of $\succ$ to constant horse/roulette lotteries) enables us to recover her entire preference relation over all horse/roulette lotteries.

Probabilistic Sophistication: A preference relation $\succ$ on $\mathcal{H}$ is said to be *probabilistically sophisticated* if there exists a *unique* subjective probability measure $\pi(.)$ over events such that for all pairs of horse/roulette lotteries $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$ and $\mathbf{H}^* = [\mathbf{R}_1^* \text{ on } E_1^*; \ldots; \mathbf{R}_{n^*}^* \text{ on } E_{n^*}^*]$:

$$\mathbf{H} \succ \mathbf{H}^* \Leftrightarrow \sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i \succ \sum_{j=1}^{n^*} \pi\left(E_j^*\right)\mathbf{R}_j^*.$$

Suppose that the agent's preferences over constant horse/roulette lotteries (her risk preferences) can be represented by a (not necessary affine) function $V(.)$. Then probabilistic sophistication (with respect to π) is equivalent to [4]

$$\mathbf{H} \succ \mathbf{H}^* \Leftrightarrow V\left(\sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i\right) > V\left(\sum_{j=1}^{n^*} \pi\left(E_j^*\right)\mathbf{R}_j^*\right).$$

If we further assume that the agent's preferences satisfy either Independence or Substitution then her preferences would admit a Subjective Expected Utility representation. Hence in their axiomatization of subjective probability, Machina and Schmeidler [8] dropped these two axioms and proposed two alternatives.

[3] Karni [5] points out that Substitution only ensures that the risk preferences are state independent. But this does not preclude the *utility* functions being state-dependent. A richer framework is required to achieve an unambiguous separation of beliefs from potentially state-dependent utility. See [5] for details.

[4] In fact, Machina and Schmeidler [8] state their definition of probabilistic sophistication in terms of the representation $V(.)$. We favor working with a definition stated purely in terms of preference as the concept of probabilistic sophistication is logically independent of whether or not the preferences admit a functional representation.

2.3. Machina and Schmeidler's alternative axioms

The first of Machina and Schmeidler's axioms weakens Substitution to hold only when $\mathbf{R}$ first-order stochastically dominates $\mathbf{R}^*$.

First-order Stochastic Dominance Preference: *For any pair of roulette lotteries* $\mathbf{R}$ *and* $\mathbf{R}^*$ *in* $\mathcal{L}$, *any horse/roulette lottery* $\mathbf{H}$ *in* $\mathcal{H}$ *and any non-null event* $E \notin \mathcal{N}$,

$$\mathbf{R} >^1 \ (\geqslant^1) \, \mathbf{R}^* \Leftrightarrow \left[\mathbf{R} \, on \, E; \ \mathbf{H} \, on \, E^c\right] \succ \left[\mathbf{R}^* \, on \, E; \ \mathbf{H} \, on \, E^c\right].$$

Any probabilistically sophisticated agent whose preferences satisfy stochastic monotonicity must satisfy this axiom. To see this, let $\mathbf{H} = [\mathbf{R}'_0 \text{ on } E, \mathbf{R}'_1 \text{ on } E_1; \ldots; \mathbf{R}'_n \text{ on } E_n]$. Suppose our agent's probability assessments are given by π. Then $\left[\mathbf{R} \text{ on } E; \ \mathbf{H} \text{ on } E^c\right]$ reduces to the roulette lottery $\pi(E)\,\mathbf{R} + \sum_{i=1}^n \pi(E_i)\,\mathbf{R}'_i$, and $\left[\mathbf{R}^* \text{ on } E; \ \mathbf{H} \text{ on } E^c\right]$ reduces to the roulette lottery $\pi(E)\,\mathbf{R}^* + \sum_{i=1}^n \pi(E_i)\,\mathbf{R}'_i$. Thus, $\mathbf{R}$ first-order stochastically dominates $\mathbf{R}^*$ if and only if $\pi(E)\,\mathbf{R} + \sum_{i=1}^n \pi(E_i)\,\mathbf{R}'_i$ first-order stochastically dominates $\pi(E)\,\mathbf{R}^* + \sum_{i=1}^n \pi(E_i)\,\mathbf{R}'_i$.

Machina and Schmeidler's final axiom allows the direct calibration of the agent's subjective probability assessments. In a sense, it is an analog of Savage's P4.

Horse/Roulette Replacement: *For any partition* $\{E_1, \ldots, E_n\}$, *if*

$$\begin{bmatrix} \delta_x & on \ E_i \\ \delta_y & on \ E_j \\ \delta_y & on \ E_k, k \neq i, j \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & on \ E_i \\ \alpha\delta_x + (1-\alpha)\,\delta_y & on \ E_j \\ \delta_y & on \ E_k, k \neq i, j \end{bmatrix}$$

for some outcomes $x \succ y$, *probability* $\alpha \in [0, 1]$ *and pair of events* E_i *and* E_j, *then*

$$\begin{bmatrix} \mathbf{R}_i & on \ E_i \\ \mathbf{R}_j & on \ E_j \\ \mathbf{R}_k & on \ E_k, k \neq i, j \end{bmatrix} \sim \begin{bmatrix} \alpha\mathbf{R}_i + (1-\alpha)\,\mathbf{R}_j & on \ E_i \\ \alpha\mathbf{R}_i + (1-\alpha)\,\mathbf{R}_j & on \ E_j \\ \mathbf{R}_k & on \ E_k, k \neq i, j \end{bmatrix} \tag{1}$$

for all roulette lotteries $\{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$.

To understand this axiom, notice that the first indifference suggests that an agent's subjective assessment of the likelihood of the event E_i relative to her subjective assessment of the likelihood of the event E_j is $\alpha/(1-\alpha)$. That is, the agent is indifferent between betting x on E_i and y on E_j or playing out the roulette lottery $\alpha\delta_x + (1-\alpha)\,\delta_y$ on the union of E_i and E_j. Like Savage's P4 axiom, Horse/Roulette Replacement stipulates that these relative likelihood assessments are invariant to the prizes directly used in the bet. But Machina and Schmeidler's axiom imposed additional invariances. First, since we are now in a Anscombe–Aumann framework, the prizes used in the bets are allowed to be roulette lotteries (the $\mathbf{R}_i$ and $\mathbf{R}_j$ in expression (1)). Second, these relative likelihood assessments are also invariant to the prizes that obtain on the states outside of E_i and E_j (the $\mathbf{R}_k$'s in expression (1)). In Anscombe–Aumann, all these invariances are derived from Independence and Substitution. In Savage, the second invariance is derived from the Sure-Thing Principle.

To illustrate how Machina and Schmeidler's argument proceeds, consider a three-event partition $\{E_1, E_2, E_3\}$ where none of the events are null. Machina and Schmeidler's continuity and Stochastic Monotonicity Axioms allow them to find unique α_1 and α_2 such that, for a pair of outcomes $x \succ y$,

$$\begin{bmatrix} \delta_x & \text{on } E_1 \\ \delta_y & \text{on } E_2 \\ \delta_y & \text{on } E_3 \end{bmatrix} \sim \begin{bmatrix} \alpha_1\delta_x + (1-\alpha_1)\,\delta_y & \text{on } E_1 \\ \alpha_1\delta_x + (1-\alpha_1)\,\delta_y & \text{on } E_2 \\ \delta_y & \text{on } E_3 \end{bmatrix}$$

and

$$\begin{bmatrix} \delta_x & \text{on } E_1 \\ \delta_x & \text{on } E_2 \\ \delta_y & \text{on } E_3 \end{bmatrix} \sim \begin{bmatrix} \alpha_2\delta_x + (1-\alpha_2)\,\delta_y & \text{on } E_1 \\ \alpha_2\delta_x + (1-\alpha_2)\,\delta_y & \text{on } E_2 \\ \alpha_2\delta_x + (1-\alpha_2)\,\delta_y & \text{on } E_3 \end{bmatrix}.$$

Now consider a horse/roulette lottery $[\mathbf{R}_1 \text{ on } \mathbf{E}_1, \mathbf{R}_2 \text{ on } \mathbf{E}_2, \mathbf{R}_3 \text{ on } \mathbf{E}_3]$ measurable with respect to this partition. Applying Horse/Roulette Replacement, we have

$$\begin{bmatrix} \mathbf{R}_1 & \text{on } E_1 \\ \mathbf{R}_2 & \text{on } E_2 \\ \mathbf{R}_3 & \text{on } E_3 \end{bmatrix} \sim \begin{bmatrix} \alpha_1\mathbf{R}_1 + (1-\alpha_1)\,\mathbf{R}_2 & \text{on } E_1 \\ \alpha_1\mathbf{R}_1 + (1-\alpha_1)\,\mathbf{R}_2 & \text{on } E_2 \\ \mathbf{R}_3 & \text{on } E_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} \alpha_2\,[\alpha_1\mathbf{R}_1 + (1-\alpha_1)\,\mathbf{R}_2] + (1-\alpha_2)\,\mathbf{R}_3 & \text{on } E_1 \\ \alpha_2\,[\alpha_1\mathbf{R}_1 + (1-\alpha_1)\,\mathbf{R}_2] + (1-\alpha_2)\,\mathbf{R}_3 & \text{on } E_2 \\ \alpha_2\,[\alpha_1\mathbf{R}_1 + (1-\alpha_1)\,\mathbf{R}_2] + (1-\alpha_2)\,\mathbf{R}_3 & \text{on } E_3 \end{bmatrix},$$

which is equal to the constant horse/roulette lottery $\alpha_1\alpha_2\mathbf{R}_1 + \alpha_2\,(1-\alpha_1)\,\mathbf{R}_2 + (1-\alpha_2)\,\mathbf{R}_3$. Since our choice of the lotteries $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ was arbitrary, we have found unique probability weights $\pi(E_1) := \alpha_1\alpha_2$, $\pi(E_2) := \alpha_2\,(1-\alpha_1)$ and $\pi(E_3) := (1-\alpha_2)$ such that, any horse/roulette lottery of the form $[\mathbf{R}'_1 \text{ on } \mathbf{E}_1, \mathbf{R}'_2 \text{ on } \mathbf{E}_2, \mathbf{R}'_3 \text{ on } \mathbf{E}_3]$ is indifferent to the constant horse/roulette lottery $\pi(E_1)\,\mathbf{R}'_1 + \pi(E_2)\,\mathbf{R}'_2 + \pi(E_3)\,\mathbf{R}'_3$; that is, the agent is Probabilistically Sophisticated. [5]

3. Probabilities via Stochastic Monotonicity

Recall that Machina and Schmeidler's Horse/Roulette Replacement axiom directly incorporates many of the invariances in probability calibration that Anscombe–Aumann derive from Independence and that Savage derives from the Sure-Thing Principle. Our aim is to use a weaker axiom to calibrate probabilities, closer in spirit to Savage's P4. We use Stochastic Monotonicity to play the role of Independence or of the Sure-Thing Principle in deriving the necessary invariances.

We first consider two implications of Stochastic Monotonicity for risk preferences. Then we extend these ideas to preferences over horse/roulette lotteries. For risk preferences, Stochastic Monotonicity may be expressed in terms of basic substitution operations on

[5] Strictly speaking it still remains to be shown that $\pi(.)$ the system of 'weights' is (finitely) additive and that the restriction of $\succ$ to constant horse/roulette lotteries admits a Mixture Continuous and Stochastically Monotonic representation. See Machina and Schmeidler [8, Appendix, p. 126] for the precise details of these remaining steps.

roulette lotteries. One basic substitution is to take an outcome in the support of a roulette lottery and change it to a worse outcome. Stochastic Monotonicity says we should prefer the lottery before this change.

Stochastic Monotonicity (I): For all $x, y \in \mathcal{X}$, all $\alpha \in (0, 1]$, and all $\mathbf{R} \in \mathcal{L}$

$$x \succ y \;\Leftrightarrow\; \alpha\delta_x + (1-\alpha)\,\mathbf{R} \succ \alpha\delta_y + (1-\alpha)\,\mathbf{R}. \tag{2}$$

Another basic substitution is to move a probability mass from a better to a worse outcome within the support of a roulette lottery. Again Stochastic Monotonicity says we should prefer the lottery before this change. Thus, if any two roulette lotteries differ only in the weight they assign to just two outcomes, the agent should prefer the lottery that puts the larger weight on the better outcome.

Stochastic Monotonicity (II): For any pair of outcomes $x, y \in \mathcal{X}$, if two roulette lotteries $\mathbf{R}, \mathbf{R}^* \in \mathcal{L}$ have the property that $\mathbf{R}(z) = \mathbf{R}^*(z)$ for all $z \notin \{x, y\}$, then

$$\mathbf{R} \succ \mathbf{R}^* \;\Leftrightarrow\; \alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**} \succ \alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**} \tag{3}$$

for all $\mathbf{R}^{**} \in \mathcal{L}$ and $\alpha \in (0, 1]$.

Notice that, in expression (3), $\mathbf{R}$ and $\mathbf{R}^*$ differ only in the probability they assign to x and y. Similarly $\alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**}$ and $\alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**}$ also differ only in the probability they assign to x and y. Furthermore $\mathbf{R}$ assigns more weight than $\mathbf{R}^*$ to the better outcome if and only $\alpha\mathbf{R} + (1-\alpha)\,\mathbf{R}^{**}$ assigns more weight than $\alpha\mathbf{R}^* + (1-\alpha)\,\mathbf{R}^{**}$ to the better outcome.

Both these Stochastic Monotonicity axioms resemble Independence in that they preserve preference ordering under the substitution of (sub-)lotteries. But these axioms only involve lotteries that differ just on two outcomes. This is a considerable weakening in that all risk preferences that satisfy Stochastic Monotonicity satisfy both these axioms, regardless of whether or not they satisfy Independence.

We now define analogs of each of these two Stochastic Monotonicity axioms to preferences over horse/roulette lotteries. For risk preferences (assuming continuity), the two Stochastic Monotonicity axioms are equivalent. But their analogs for horse/roulette lotteries are not equivalent, so we will impose them separately.

The following is an analog of Stochastic Monotonicity (I) for horse/roulette lotteries. It restricts Substitution to apply only to the substitutions of degenerate roulette lotteries.

Degenerate Substitution: *For any pair of outcomes* $x, y \in \mathcal{X}$, *any horse/roulette lottery* $\mathbf{H} \in \mathcal{L}$ *and any non-null event* E,

$$x \succ y \Leftrightarrow \left[\delta_x \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right] \succ \left[\delta_y \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right]. \tag{4}$$

To see the analogy to Stochastic Monotonicity (I), let $\mathbf{H} = [\mathbf{R}_0 \text{ on } E, \mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$. Suppose the agent's probability assessments are given by π then the terms $[\delta_x$ on $E;\ \mathbf{H}$ on $E^{\mathrm{c}}]$ and $[\delta_y$ on $E;\ \mathbf{H}$ on $E^{\mathrm{c}}]$ on the right-hand side of expression (4) reduce to the roulette lotteries $\pi(E)\left[\delta_x\right] + \sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i$ and $\pi(E)\left[\delta_y\right] + \sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i$, similar to the terms on the right-hand side of expression (2). Therefore, if a Probabilistically Sophisticated agent satisfies Stochastic Monotonicity (I), she must satisfy Degenerate Substitution.

The following is an analog of Stochastic Monotonicity (II) for horse/roulette lotteries. It restricts Independence to apply only if the two horse/roulette lotteries in question have the property that, within each state, the roulette lotteries differ only in the probability they assign to two outcomes, and these two outcomes on which they differ are the same two outcomes in every state.

Two-Outcome Independence: *For any pair of outcomes* $x, y \in \mathcal{X}$, *if two horse/roulette lotteries* $\mathbf{H}, \mathbf{H}^* \in \mathcal{H}$ *have the property that* $\mathbf{H}(s)(z) = \mathbf{H}^*(s)(z)$ *for all* $z \notin \{x, y\}$ *and all* $s \in \mathcal{S}$, *then*

$$\mathbf{H} \succ \mathbf{H}^* \;\Leftrightarrow\; \alpha\mathbf{H} + (1-\alpha)\,\mathbf{H}^{**} \succ \alpha\mathbf{H}^* + (1-\alpha)\,\mathbf{H}^{**} \tag{5}$$

for all $\mathbf{H}^{**} \in \mathcal{H}$ *and* $\alpha \in (0, 1]$.

To see the analogy to Stochastic Monotonicity (II), suppose the agent's probability assessments are given by π, and let $\mathbf{R}$, $\mathbf{R}^*$ and $\mathbf{R}^{**}$ in expression (3) be the reductions of $\mathbf{H}$, $\mathbf{H}^*$ and $\mathbf{H}^{**}$ from expression (5). For example, if $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$ then $\mathbf{R} = \sum_{i=1}^{n} \pi(E_i)\,\mathbf{R}_i$. Since, in each state, $\mathbf{H}(s)$ and $\mathbf{H}^*(s)$ differ only in the probability they assign to x and y, their reductions $\mathbf{R}$ and $\mathbf{R}^*$ differ only in the probability they assign to x and y. Therefore, if a Probabilistically Sophisticated agent satisfies Stochastic Monotonicity (II), she must satisfy Two-Outcome Independence.

Our final axiom is an analog of Savage's P4. In Savage's axiomatization, an agent reveals that she thinks the event A is more likely than the event B if she would rather bet on A than B. In an Anscombe–Aumann framework, we have the additional device that we can compare bets on subjective events with bets on objective probabilities. This allows us to calibrate the likelihood of events. Savage's P4 simply says that his rankings of events are invariant to the stakes of the bets. The following axiom simply says that our calibrations of events are invariant to the stakes of the bets.

Betting Neutrality: *For any four outcomes* $x, y, x', y' \in \mathcal{X}$, *any* $p \in [0, 1]$ *and any event* $E \in \mathcal{E}$, *if* $\delta_x \succ \delta_y$ *and* $\delta_{x'} \succ \delta_{y'}$ *then*

$$\begin{bmatrix} \delta_x & \text{on } E \\ \delta_y & \text{on } E^{\mathrm{c}} \end{bmatrix} \succ p\delta_x + (1-p)\,\delta_y \Rightarrow \begin{bmatrix} \delta_{x'} & \text{on } E \\ \delta_{y'} & \text{on } E^{\mathrm{c}} \end{bmatrix} \succ p\delta_{x'} + (1-p)\,\delta_{y'}.$$

This axiom is weaker than Horse/Roulette Replacement. In particular, by itself, Betting Neutrality is not strong enough to yield the following immediate implications of Horse/Roulette Replacement. First, the new axiom does not by itself imply that the calibration of *conditional* events is invariant to the stakes of the bet. That is, Betting Neutrality does not on its own imply that

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } B \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \succ \begin{bmatrix} p\delta_x + (1-p)\,\delta_y & \text{on } (A \cup B) \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \delta_{x'} & \text{on } A \\ \delta_{y'} & \text{on } B \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \succ \begin{bmatrix} p\delta_{x'} + (1-p)\,\delta_{y'} & \text{on } (A \cup B) \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$

for all outcomes $x, y, x', y' \in \mathcal{X}$, $\delta_x \succ \delta_y$ and $\delta_{x'} \succ \delta_{y'}$, any $p \in [0, 1]$, any horse/roulette lotteries $\mathbf{H} \in \mathcal{H}$, and any disjoint events $A, B \in \mathcal{E}$.

Second, the new axiom does not by itself imply that the calibration of conditional events is invariant to what would arise outside the conditioning event. That is, Betting Neutrality does not on its own imply that

$$\begin{bmatrix} \delta_{x} & \text{on } A \\ \delta_{y} & \text{on } B \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \succ \begin{bmatrix} p\delta_{x} + (1-p)\,\delta_{y} & \text{on } (A \cup B) \\ \mathbf{H}(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \delta_{x} & \text{on } A \\ \delta_{y} & \text{on } B \\ \mathbf{H}'(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \succ \begin{bmatrix} p\delta_{x} + (1-p)\,\delta_{y} & \text{on } (A \cup B) \\ \mathbf{H}'(s) & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$

for all outcomes $x, y \in \mathcal{X}$, $\delta_x \succ \delta_y$, any $p \in [0, 1]$, any horse/roulette lotteries $\mathbf{H}, \mathbf{H}' \in \mathcal{H}$, and any disjoint events $A, B \in \mathcal{E}$.[6]

Third, the new axiom does not even by itself imply that the calibration of *unconditional* events is the same if the calibration is obtained by willingness to bet *on* an event versus willingness to bet *against* the event. That is, Betting Neutrality does not on its own imply that

$$\begin{bmatrix} \delta_{x} & \text{on } E \\ \delta_{y} & \text{on } E^{\mathrm{c}} \end{bmatrix} \succ p\delta_x + (1-p)\,\delta_y \Rightarrow p\delta_y + (1-p)\,\delta_x \succ \begin{bmatrix} \delta_{y} & \text{on } E \\ \delta_{x} & \text{on } E^{\mathrm{c}} \end{bmatrix}$$

for all outcomes $x, y \in \mathcal{X}$, $\delta_x \succ \delta_y$, any $p \in [0, 1]$ and any event $E \in \mathcal{E}$.[7]

Although Betting Neutrality is weak, in the presence of our Stochastic Monotonicity axioms, it is sufficient for Probabilistic Sophistication. This is our main result.

Theorem. *The following two statements are equivalent*:

(a) *The preference relation $\succ$ on $\mathcal{H}$ satisfies Ordering*, *Archimedean*, *Non-degeneracy*, *Degenerate Substitution*, *Two-Outcome Independence and Betting Neutrality*.
(b) *There exists a unique finite-additive probability measure π defined over subsets of $\mathcal{S}$ and a non-constant*, *Mixture Continuous*, *Stochastically Monotonic function V on $\mathcal{L}$*, *such that for all pairs of horse/roulette lotteries $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$*

[6] For example, preferences can satisfy Betting Neutrality and still be consistent with the Ellsberg three-color paradox. In particular, most of the standard models of preference that violate probabilistic sophistication still satisfy Betting Neutrality; for example, Choquet Expected Utility [10], Multiple-Prior preferences [3], and more general biseparable preferences [4].

[7] For preferences that satisfy Betting Neutrality but do not satisfy this property, see the example used in Section 4 to show that Two-Outcome Independence is essential.

and $\mathbf{H}^* = [\mathbf{R}_1^*$ *on* $E_1^*; \ldots; \mathbf{R}_{n^*}^*$ *on* $E_{n^*}]$ *in* $\mathcal{H}$

$$\mathbf{H} \succ \mathbf{H}^* \Leftrightarrow V\left(\sum_{i=1}^{n} \pi\left(E_i\right) \mathbf{R}_i\right) > V\left(\sum_{j=1}^{n^*} \pi\left(E_j^*\right) \mathbf{R}_j^*\right).$$

The details of the proof are in the appendix, but an intuition for the sufficiency of the axioms is as follows. It is enough to show that our axioms imply Machina and Schmeidler's axioms. Taking Order, Archimedean and Non-Degeneracy as given, our two Stochastic Monotonicity axioms (Degenerate Substitution and Two-Outcome Independence) imply Machina and Schmeidler's Stochastic Monotonicity axiom (First-order Stochastic Dominance Preference). The harder step is to show that these Stochastic Monotonicity axioms are sufficient to extend our weak calibration axiom (Betting Neutrality) to obtain Machina and Schmeidler's much stronger calibration axiom (Horse/Roulette Replacement).

An analogy may be useful here. Savage's P4, like our Betting Neutrality, does not imply all the invariances on conditional likelihoods entailed in Horse/Roulette Replacement. But Savage's Sure-Thing Principle (his axiom P2) allows him to move from invariant unconditional likelihoods to invariant conditional likelihoods. In Anscombe–Aumann, the Independence Axiom gives the Sure-Thing Principle for free. These axioms preserve preference ordering under the substitution of (sub-)lotteries. Our Stochastic Monotonicity axioms are not as strong as Independence or the Sure-Thing Principle in that they do not apply to the substitution of general (sub-)lotteries. But they allow enough substitution to extend Betting Neutrality to Horse/Roulette Replacement. In particular, Two-Outcome Independence is sufficient to imply a 'two-outcome' version of the Sure-Thing Principle. Two outcomes turn out to be enough.

4. Independence of the axioms

We now provide examples to show that each of our new axioms, Degenerate Substitution, Two-Outcome Independence and Betting Neutrality is essential. Each example uses a two-element state space $\mathcal{S} = \{s_1, s_2\}$, and outcomes $\mathcal{X} = [0, 1]$.

The examples we use to show that Degenerate Substitution and Betting Neutrality are essential, each use preferences that can be represented by the following State-Dependent Expected Utility function:

$$V(\mathbf{H}) = \sum_{x \in \mathcal{X}} u_1(x)\, \mathbf{H}(s_1)(x) + \sum_{x \in \mathcal{X}} u_2(x)\, \mathbf{H}(s_2)(x).$$

It is easy to check that these preferences satisfy Ordering, Archimedean and Two-Outcome Independence. (In fact, these preferences satisfy full Independence.)

Degenerate Substitution: To show that Degenerate Substitution is essential, consider the case where $u_1(x) = 2x$ and $u_2(x) = -x$; that is, the agent prefers high outcomes in state s_1 and prefers low outcomes in state s_2 where the first preference is 'twice as strong' as the second. Clearly, these preferences are not Probabilistically Sophisticated. These preferences satisfy Non-Degeneracy since, for any outcome x, we have $V(\delta_x) = x$. Thus $x \succ y$ if (and only if) $x > y$. To show that they satisfy Betting Neutrality, notice that for any $x > y$, and any $\alpha \in [0, 1]$

$$V\left(\left[\delta_x \text{ on } s_1; \delta_y \text{ on } s_2\right]\right) = 2x - y > \alpha x + (1-\alpha)\, y = V\left(\alpha\delta_x + (1-\alpha)\,\delta_y\right),$$
$$V\left(\left[\delta_y \text{ on } s_1; \delta_x \text{ on } s_1\right]\right) = 2y - x < \alpha x + (1-\alpha)\, y = V\left(\alpha\delta_x + (1-\alpha)\,\delta_y\right)$$

so Betting Neutrality holds trivially.

It remains to show that these preferences do not satisfy Degenerate Substitution. For any $x > y$, we have $V(\delta_x) = x > y = V\left(\delta_y\right)$. But, for any $\mathbf{R}$, $V\left(\left[\delta_x \text{ on } s_2;\ \mathbf{R} \text{ on } s_1\right]\right) - V\left(\left[\delta_y \text{ on } s_2;\ \mathbf{R} \text{ on } s_1\right]\right) = (-x) - (-y) < 0$, contradicting Degenerate Substitution.

Betting Neutrality: To show that Betting Neutrality is essential, now consider the case where both u_1 and u_2 are strictly increasing so they order the outcomes the same way, but assume that u_1 is not a positive affine transformation of u_2 so the risk preferences differ in the two states. Once again, these preferences are not Probabilistically Sophisticated but they do satisfy Non-Degeneracy since for any outcome $V(\delta_x) = u_1(x) + u_2(x)$, thus $x \succ y$ if (and only if) $x > y$. Degenerate Substitution also holds since for any $x > y$, any state s_i and any $\mathbf{R}$, $V\left(\left[\delta_x \text{ on } s_i;\ \mathbf{R} \text{ on } s_j\right]\right) - V\left(\left[\delta_y \text{ on } s_i;\ \mathbf{R} \text{ on } s_j\right]\right) = u_i(x) - u_i(y) > 0$.

To see that they fail to satisfy Betting Neutrality first notice for $u_2(.)$ not to be an affine transformation of $u_2(.)$ there must exist three outcomes $x > y > z$ for which

$$\frac{u_1(x) - u_1(y)}{u_2(x) - u_2(y)} \neq \frac{u_1(x) - u_1(z)}{u_2(x) - u_2(z)},$$

Therefore, fix any $x > y > z$ and consider the horse/roulette lotteries $[\delta_x \text{ on } s_1;\ \delta_y \text{ on } s_2]$ and $[\delta_x \text{ on } s_1;\ \delta_z \text{ on } s_2]$. If Betting Neutrality holds, then

$$V\left(\left[\delta_x \text{ on } s_1;\ \delta_y \text{ on } s_2\right]\right) = V\left(\alpha\delta_x + (1-\alpha)\,\delta_y\right)$$
$$\Leftrightarrow V\left(\left[\delta_x \text{ on } s_1;\ \delta_z \text{ on } s_2\right]\right) = V\left(\alpha\delta_x + (1-\alpha)\,\delta_z\right).$$

Substituting in from the definition of $V(\mathbf{H})$ these two equalities imply,

$$\frac{u_1(x) - u_1(y)}{u_2(x) - u_2(y)} = \frac{\alpha}{1-\alpha} = \frac{u_1(x) - u_1(z)}{u_2(x) - u_2(z)}.$$

Since our choice of x, y and z was arbitrary, u_1 must be a positive affine transformation of u_2, a contradiction.

Two-Outcome Independence: To show that Two-Outcome Independence is essential, consider preferences represented by a certainty-equivalent function $V : \mathcal{H} \to [0, 1]$, defined by

$$V(\mathbf{H}) = \begin{cases} \frac{2}{3}E[\mathbf{H}(s_1)] + \frac{1}{3}E[\mathbf{H}(s_2)] & \text{if } E[\mathbf{H}(s_1)] \leqslant E[\mathbf{H}(s_2)], \\ \frac{1}{3}E[\mathbf{H}(s_1)] + \frac{2}{3}E[\mathbf{H}(s_2)] & \text{if } E[\mathbf{H}(s_1)] > E[\mathbf{H}(s_2)], \end{cases} \tag{6}$$

where $E[\mathbf{R}]$ is the expected value of $\mathbf{R}$.

It is easy to check that these preferences satisfy Ordering, Archimedean, Non-Degeneracy, and Degenerate Substitution. (Indeed, these preferences satisfy full Substitution.) Clearly these preferences are not probabilistically sophisticated. To show that these preferences satisfy Betting Neutrality, fix a pair of outcomes $x > y$ and a state s_i, and consider the horse/roulette lottery $\left[\delta_x \text{ on } s_i;\ \delta_y \text{ on } s_j\right]$. By construction, $x > V(\left[\delta_x \text{ on } s_i;\ \delta_y \text{ on } s_j\right]) > y$. It follows from Eq. (6) that $V([\delta_x \text{ on } s_i;\ \delta_y \text{ on } s_j]) = (x + 2y)/3$, therefore $\left[\delta_x \text{ on } s_i;\ \delta_y \text{ on } s_j\right] \sim \left(\frac{1}{3}\right)\delta_x + \left(\frac{2}{3}\right)\delta_y$. Since the choice of $x > y$ was arbitrary, we have, for all $w > z$

$$\left[\delta_w \text{ on } s_i;\ \delta_z \text{ on } s_j\right] \succ p\delta_w + (1-p)\,\delta_z \Leftrightarrow p < \tfrac{1}{3}.$$

And since s_i was also arbitrary (and there are only two states), this is equivalent to betting neutrality.

It remains to show that these preferences do not satisfy Two-Outcome Independence. We show this by contradiction. Since s_i was arbitrary in the argument above, we know that for $x > y$, $\left[\delta_x \text{ on } s_1;\ \delta_y \text{ on } s_2\right] \sim \left[\delta_x \text{ on } s_2;\ \delta_y \text{ on } s_1\right] \sim \left(\frac{1}{3}\right)\delta_x + \left(\frac{2}{3}\right)\delta_y$. Thus, applying Two-Outcome Independence twice implies that

$$\frac{1}{2}\begin{bmatrix}\delta_x & \text{on } s_1\\ \delta_y & \text{on } s_2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\delta_y & \text{on } s_1\\ \delta_x & \text{on } s_2\end{bmatrix} \sim \frac{1}{2}\left[\frac{1}{3}\delta_x + \frac{2}{3}\delta_y\right] + \frac{1}{2}\begin{bmatrix}\delta_y & \text{on } s_1\\ \delta_x & \text{on } s_2\end{bmatrix}$$

$$\sim \frac{1}{2}\left[\frac{1}{3}\delta_x + \frac{2}{3}\delta_y\right] + \frac{1}{2}\left[\frac{1}{3}\delta_x + \frac{2}{3}\delta_y\right]$$
$$= \frac{1}{3}\delta_x + \frac{2}{3}\delta_y.$$

But,

$$\frac{1}{2}\begin{bmatrix}\delta_x & \text{on } s_1\\ \delta_y & \text{on } s_2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\delta_y & \text{on } s_1\\ \delta_x & \text{on } s_2\end{bmatrix} = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y,$$

which is a contradiction.

Appendix. Proof

The proof of (b)⇒(a) (necessity of the axioms) is straightforward. The following is a proof of (a)⇒(b) (sufficiency of the axioms). We proceed in three steps. In Step 1, we show that given Ordering, Archimedean and Non-Degeneracy, our new stochastic monotonicity axioms, Degenerate Substitution and Two-Outcome Independence imply Machina–Schmeidler's axiom First-order Stochastic Dominance Preference. Step 2 is more interesting. In Step 2, we add our calibration axiom, Betting Neutrality, and derive Machina–Schmeidler's axiom, Horse/Roulette Replacement. In Step 3, we derive a utility representation for the preference relation restricted to the set of constant acts, $V : \mathcal{L} \to \mathbb{R}$. The combination of these steps is sufficient because one can now proceed as Machina and Schmeidler do to extend the function V to the whole domain of horse/roulette lotteries. That is, for arbitrary horse/roulette lottery $\mathbf{H} = [\mathbf{R}_1 \text{ on } E_1; \ldots; \mathbf{R}_n \text{ on } E_n]$, we can set $V(\mathbf{H}) = V\left(\sum_{i=1}^{n} \pi(E_i)\, \mathbf{R}_i\right)$.

Step 1: The preference relation $\succ$ satisfies First-order Stochastic Dominance Preference.

Fix a horse/roulette lottery $\mathbf{H} \in \mathcal{H}$, a pair of roulette lotteries $\mathbf{R}$ and $\mathbf{R}^*$ in $\mathcal{L}$, and non-null event E. In order to prove the result we state without proof the following well-known fact.

Fact 1. *If* $\mathbf{R} \geqslant_1 \mathbf{R}^*$ *then there exists a finite sequence of tuples* $\left(\alpha_n, x_n, y_n, \mathbf{R}_n, \mathbf{R}'_n\right)_{n=1}^{N} \in (0,1] \times \mathcal{X}^2 \times \mathcal{L}$, *such that* $\mathbf{R} = \mathbf{R}_1$, $\mathbf{R}^* = \alpha_N \delta_{y_N} + (1-\alpha_N)\, \mathbf{R}'_N$ *and for each* $n = 1, \ldots, N-1$, $x_n \succsim y_n$,

$$\begin{aligned} \mathbf{R}_n &= \alpha_n \delta_{x_n} + (1-\alpha_n)\, \mathbf{R}'_n, \\ \mathbf{R}_{n+1} &= \alpha_n \delta_{y_n} + (1-\alpha_n)\, \mathbf{R}'_n. \end{aligned}$$

If $\mathbf{R} >_1 \mathbf{R}^*$ *then for at least one* $n \in \{1, \ldots, N\}$, $x_n \succ y_n$.

That is, Fact 1 says that if $\mathbf{R} \geqslant_1 \mathbf{R}^*$, then $\mathbf{R}^*$ may be obtained from $\mathbf{R}$ by a finite sequence of operations of taking probability mass off an outcome in a roulette lottery and placing it on a no more desirable outcome. And if $\mathbf{R} >_1 \mathbf{R}^*$ then one of these operations must involve moving a probability mass from an outcome onto a strictly less desirable outcome.

Applying first Degenerate Substitution and then Two-Outcome Independence, we have

$$\begin{aligned} x_n \succ (\succsim)\, y_n &\Rightarrow \left[\delta_{x_n} \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right] \succ (\succsim) \left[\delta_{y_n} \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right] \\ &\Rightarrow \left[\alpha_n \delta_{x_n} + (1-\alpha_n)\, \mathbf{R}'_n \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right] \\ &\quad \succ (\succsim) \left[\alpha_n \delta_{y_n} + (1-\alpha_n)\, \mathbf{R}'_n \text{ on } E;\ \mathbf{H} \text{ on } E^{\mathrm{c}}\right]. \end{aligned}$$

Given Fact 1, the result follows from transitivity.

Step 2: The preference relation $\succ$ satisfies Horse/Roulette Replacement.

The preference relation satisfies Ordering, Archimedean and (from Step 1) First-order Stochastic Dominance Preference. It is a standard result in utility theory that these axioms imply that $\succ$ satisfies the following property:

Fact 2 (*Two-Outcome Mixture Solvability*). *For any pair of outcomes* $x, y \in \mathcal{X}$, *any act* $\mathbf{H} \in \mathcal{H}$, *and any non-null event* $E \notin \mathcal{N}$, *if*

$$\begin{bmatrix} \delta_x & s \in E \\ \mathbf{H}(s) & s \notin E \end{bmatrix} \succ \mathbf{H} \succ \begin{bmatrix} \delta_y & s \in E \\ \mathbf{H}(s) & s \notin E \end{bmatrix}$$

then there exists a unique α *in* $[0, 1]$, *such that*

$$\begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & s \in E \\ \mathbf{H}(s) & s \notin E \end{bmatrix} \sim \mathbf{H}.$$

For a proof see [2, Lemma 3.1, p. 33].

To show that Horse/Roulette Replacement obtains, we have to show that probability assessments satisfy a series of invariance conditions. We first show that the probability assessment does not depend on whether the better outcome is received on A or A^c.

Lemma 1. *For any pair of outcomes x and y, any non-null event $A \notin \mathcal{N}$, and any α in* $[0, 1]$

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } A^c \end{bmatrix} \sim \alpha\delta_x + (1-\alpha)\,\delta_y \Rightarrow \begin{bmatrix} \delta_y & \text{on } A \\ \delta_x & \text{on } A^c \end{bmatrix} \sim \alpha\delta_y + (1-\alpha)\,\delta_x.$$

Proof. If $\delta_x \sim \delta_y$ then the result follows immediately from Degenerate Substitution and Two-Outcome Independence. So, suppose (wlog) $\delta_x \succ \delta_y$. By Property 2 (Two-Outcome Mixture Solvability) there exists unique $\alpha, \beta \in [0, 1]$ for which

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } A^c \end{bmatrix} \sim \alpha\delta_x + (1-\alpha)\,\delta_y \text{ and } \begin{bmatrix} \delta_y & \text{on } A \\ \delta_x & \text{on } A^c \end{bmatrix} \sim \beta\delta_y + (1-\beta)\,\delta_x.$$

Now by construction

$$\frac{1}{2}\delta_x + \frac{1}{2}\delta_y = \frac{1}{2}\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } A^c \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \delta_y & \text{on } A \\ \delta_x & \text{on } A^c \end{bmatrix}.$$

And so by Two-Outcome Independence,

$$\frac{1}{2}\delta_x + \frac{1}{2}\delta_y \sim \left(\frac{\alpha+1-\beta}{2}\right)\delta_x + \left(\frac{\beta+1-\alpha}{2}\right)\delta_y.$$

Equating coefficients (which follows from First-order Stochastic Dominance Preference via step 1), we obtain $\alpha = \beta$, as required. □

We next show that conditional probability assessments do not depend on the degenerate prizes involved in the bets.

Lemma 2. *For any four outcomes w, x, y and z, and any pair of disjoint events A, B, and any α in* $[0, 1]$, *if either $\delta_x \succ \delta_y$ or $\delta_y \succ \delta_x$ then*

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } B \\ \delta_y & \text{on } (A \cup B)^c \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & \text{on } A \cup B \\ \delta_y & \text{on } (A \cup B)^c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \delta_w & \text{on } A \\ \delta_z & \text{on } B \\ \delta_z & \text{on } (A \cup B)^c \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_w + (1-\alpha)\,\delta_z & \text{on } A \cup B \\ \delta_z & \text{on } (A \cup B)^c \end{bmatrix}.$$

Proof. Whether $\delta_x \succ \delta_y$ or $\delta_y \succ \delta_x$, if $\delta_w \sim \delta_z$ then the result follows immediately from Degenerate Substitution and Two-Outcome Independence. So consider the case where $\delta_x \succ \delta_y$ and $\delta_w \succ \delta_z$. The argument for the other cases are similar after applying Lemma 1. If $(A \cup B)^c$ is null, then the result follows immediately from Betting Neutrality. If $A \cup B$ is null, then it holds trivially. If A is null and B is not, then from Degenerate Substitution and

Two-Outcome Independence $\alpha = 0$ in both lines. Similarly, if B is null and A is not, then $\alpha = 1$ in both lines.

Assuming A, B and $(A \cup B)^c$ are non-null, by Degenerate Substitution, Fact 2 (Two-Outcome Mixture Solvability) and Betting Neutrality, there exist unique $\alpha_A, \alpha_{A\cup B} \in (0, 1)$ such that

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } A^c \end{bmatrix} \sim \alpha_A \delta_x + (1-\alpha_A)\,\delta_y,$$

$$\begin{bmatrix} \delta_w & \text{on } A \\ \delta_z & \text{on } A^c \end{bmatrix} \sim \alpha_A \delta_w + (1-\alpha_A)\,\delta_z,$$

$$\begin{bmatrix} \delta_x & \text{on } A\cup B \\ \delta_y & \text{on } (A\cup B)^c \end{bmatrix} \sim \alpha_{A\cup B} \delta_x + (1-\alpha_{A\cup B})\,\delta_y,$$

$$\begin{bmatrix} \delta_w & \text{on } A\cup B \\ \delta_z & \text{on } (A\cup B)^c \end{bmatrix} \sim \alpha_{A\cup B} \delta_w + (1-\alpha_{A\cup B})\,\delta_z.$$

And by Two-Outcome Mixture Solvability, there exist unique $\alpha, \beta \in [0, 1]$ such that

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } B \\ \delta_y & \text{on } (A\cup B)^c \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & \text{on } A\cup B \\ \delta_y & \text{on } (A\cup B)^c \end{bmatrix}$$

and

$$\begin{bmatrix} \delta_w & \text{on } A \\ \delta_z & \text{on } B \\ \delta_z & \text{on } (A\cup B)^c \end{bmatrix} \sim \begin{bmatrix} \beta\delta_w + (1-\beta)\,\delta_z & \text{on } A\cup B \\ \delta_z & \text{on } (A\cup B)^c \end{bmatrix}.$$

We need to show that $\alpha = \beta$. By construction

$$\begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & \text{on } A\cup B \\ \delta_y & \text{on } (A\cup B)^c \end{bmatrix} = \alpha \begin{bmatrix} \delta_x & \text{on } A\cup B \\ \delta_y & \text{on } (A\cup B)^c \end{bmatrix} + (1-\alpha)\,\delta_y$$

and

$$\begin{bmatrix} \beta\delta_w + (1-\beta)\,\delta_z & \text{on } A\cup B \\ \delta_z & \text{on } (A\cup B)^c \end{bmatrix} = \beta \begin{bmatrix} \delta_w & \text{on } A\cup B \\ \delta_z & \text{on } (A\cup B)^c \end{bmatrix} + (1-\beta)\,\delta_z.$$

So by Two-Outcome Independence and Order we have,

$$\alpha_A \delta_x + (1-\alpha_A)\,\delta_y \sim \alpha \times \alpha_{A\cup B}\delta_x + (1-\alpha \times \alpha_{A\cup B})\,\delta_y$$

and

$$\alpha_A \delta_w + (1-\alpha_A)\,\delta_z \sim \beta \times \alpha_{A\cup B}\delta_w + (1-\beta \times \alpha_{A\cup B})\,\delta_z.$$

Equating coefficients (which follows from First-order Stochastic Dominance Preference via step 1), we obtain $\alpha_A = \alpha \times \alpha_{A\cup B}$ and $\alpha_A = \beta \times \alpha_{A\cup B} \Rightarrow \alpha = \beta$, as required. □

Next, we show that conditional probability assessments do not depend on the prizes that obtain outside the conditioning event. That is, we require that for any $x \succ y$,

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } B \\ \delta_y & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_x + (1-\alpha)\,\delta_y & \text{on } A \cup B \\ \delta_y & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \delta_w & \text{on } A \\ \delta_z & \text{on } B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_w + (1-\alpha)\,\delta_z & \text{on } A \cup B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix}$$

for any pair of outcomes w and z in $\mathcal{X}$, and any horse/roulette lottery $\mathbf{H}$ in $\mathcal{X}$.

To obtain this invariance, it is sufficient to prove the following 'two-outcome' restriction of the classic Sure-Thing Principle.

Lemma 3 (*Two-Outcome Sure Thing Principle*). *If $\succ$ satisfies Ordering and Two-Outcome Independence then it satisfies the following property. For any pair of outcomes $x, y \in \mathcal{X}$, if two horse/roulette lotteries $\mathbf{H}, \mathbf{H}^* \in \mathcal{H}$ have the property that $\mathbf{H}(s)(z) = \mathbf{H}^*(s)(z)$ for all $z \notin \{x, y\}$ and all $s \in \mathcal{S}$, then for any event $E \subset \mathcal{S}$, and any pair of horse/roulette lotteries $\mathbf{H}^{**}$ and $\mathbf{H}^{***}$ in $\mathcal{H}$:*

$$\left[\mathbf{H} \textit{ on } E;\ \mathbf{H}^{**} \textit{ on } E^{\mathrm{c}}\right] \succ \left[\mathbf{H}^* \textit{ on } E;\ \mathbf{H}^{**} \textit{ on } E^{\mathrm{c}}\right]$$
$$\Rightarrow \left[\mathbf{H} \textit{ on } E;\ \mathbf{H}^{***} \textit{ on } E^{\mathrm{c}}\right] \succ \left[\mathbf{H}^* \textit{ on } E;\ \mathbf{H}^{***} \textit{ on } E^{\mathrm{c}}\right].$$

Proof. Suppose not. That is, there exists a pair of outcomes $\{x, y\} \subset \mathcal{X}$, a pair of horse/roulette lotteries $\mathbf{H}, \mathbf{H}^* \in \mathcal{H}$ with the property that $\mathbf{H}(s)(z) = \mathbf{H}^*(s)(z)$ for all $z \notin \{x, y\}$ and all $s \in \mathcal{S}$, an event $E \subset \mathcal{S}$, and a pair of horse/roulette lotteries $\mathbf{H}^{**}$ and $\mathbf{H}^{***}$ in $\mathcal{H}$ such that $\left[\mathbf{H} \text{ on } E;\ \mathbf{H}^{**} \text{ on } E^{\mathrm{c}}\right] \succ \left[\mathbf{H}^* \text{ on } E;\ \mathbf{H}^{**} \text{ on } E^{\mathrm{c}}\right]$ but $\left[\mathbf{H}^* \text{ on } E;\ \mathbf{H}^{***} \text{ on } E^{\mathrm{c}}\right] \succsim \left[\mathbf{H} \text{ on } E;\ \mathbf{H}^{***} \text{ on } E^{\mathrm{c}}\right]$. Then, by applying Two-Outcome Independence twice we obtain

$$\frac{1}{2}\begin{bmatrix} \mathbf{H}(s) & s \in E \\ \mathbf{H}^{**}(s) & s \notin E \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{***}(s) & s \notin E \end{bmatrix}$$
$$\succ \frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{**}(s) & s \notin E \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{***}(s) & s \notin E \end{bmatrix}$$

and

$$\frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{**}(s) & s \notin E \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{***}(s) & s \notin E \end{bmatrix}$$
$$\succ \frac{1}{2}\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \mathbf{H}^{**}(s) & s \notin E \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{H}(s) & s \in E \\ \mathbf{H}^{***}(s) & s \notin E \end{bmatrix}.$$

That is,

$$\begin{bmatrix} \frac{1}{2}\mathbf{H}(s) + \frac{1}{2}\mathbf{H}^*(s) & s \in E \\ \frac{1}{2}\mathbf{H}^{**}(s) + \frac{1}{2}\mathbf{H}^{***}(s) & s \notin E \end{bmatrix} \succ \begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \frac{1}{2}\mathbf{H}^{**}(s) + \frac{1}{2}\mathbf{H}^{***}(s) & s \notin E \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{H}^*(s) & s \in E \\ \frac{1}{2}\mathbf{H}^{**}(s) + \frac{1}{2}\mathbf{H}^{***}(s) & s \notin E \end{bmatrix} \succ \begin{bmatrix} \frac{1}{2}\mathbf{H}(s) + \frac{1}{2}\mathbf{H}^*(s) & s \in E \\ \frac{1}{2}\mathbf{H}^{**}(s) + \frac{1}{2}\mathbf{H}^{***}(s) & s \notin E \end{bmatrix},$$

a contradiction. □

Finally, to establish that $\succ$ satisfies Horse/Roulette Replacement, we have to show that conditional probability assessments do not depend on the prizes being non-degenerate outcomes. This follows from the fact that for any pair of roulette lotteries $\mathbf{R}$ and $\mathbf{R}^*$, there exist a finite list of probability weights $\langle \lambda_1, \ldots, \lambda_m \rangle$, and two finite lists of (not necessarily distinct) outcomes $\langle x_1, \ldots, x_m \rangle$ and $\langle y_1, \ldots, y_m \rangle$, for which $\mathbf{R} = \sum_{i=1}^m \lambda_i \delta_{x_i}$ and $\mathbf{R}^* = \sum_{i=1}^m \lambda_i \delta_{y_i}$. Hence we have by repeated applications of the results above, for any $x \succ y$,

$$\begin{bmatrix} \delta_x & \text{on } A \\ \delta_y & \text{on } B \\ \delta_y & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix} \sim \begin{bmatrix} \alpha\delta_x + (1-\alpha)\delta_y & \text{on } A \cup B \\ \delta_y & \text{on } (A \cup B)^{\mathrm{c}} \end{bmatrix}$$

implies

$$\begin{aligned}
\begin{bmatrix} \mathbf{R} & \text{on } A \\ \mathbf{R}^* & \text{on } B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} &= \sum_{i=1}^m \lambda_i \begin{bmatrix} \delta_{x_i} & \text{on } A \\ \delta_{y_i} & \text{on } B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} \\
&\sim \lambda_1 \begin{bmatrix} \alpha\delta_{x_1} + (1-\alpha)\delta_{y_1} & \text{on } A \cup B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} + \sum_{i=2}^m \lambda_i \begin{bmatrix} \delta_{x_i} & \text{on } A \\ \delta_{y_i} & \text{on } B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} \\
&\sim \ldots \sim \sum_{i=1}^m \lambda_i \begin{bmatrix} \alpha\delta_{x_i} + (1-\alpha)\delta_{y_i} & \text{on } A \cup B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} \\
&= \begin{bmatrix} \alpha \sum_{i=1}^m \lambda_i \delta_{x_i} + (1-\alpha)\sum_{i=1}^m \lambda_i \delta_{y_i} & \text{on } A \cup B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix} \\
&= \begin{bmatrix} \alpha\mathbf{R} + (1-\alpha)\mathbf{R}^* & \text{on } A \cup B \\ \mathbf{H}(s) & s \notin (A \cup B) \end{bmatrix}
\end{aligned}$$

for every $\mathbf{H}^*$ in $\mathcal{H}$. That is, $\succ$ satisfies Horse/Roulette Replacement.

Step 3: Constructing the utility representation on $\succ$ restricted to $\mathcal{L}$, the set of constant acts.

If there exists best and worst outcomes then take x to be from the indifference class of best outcomes and y to be from the indifference class of worst outcomes. Set $V(\delta_x) := 1$ and $V(\delta_y) := 0$. For any $L \in \mathcal{L}$, set $V(L) := \beta$, where, by Property 2 (Two-Outcome Mixture Solvability), β is the unique solution to $\beta\delta_x + (1-\beta)\delta_y \sim \mathbf{R}$.

Hence for any pair of pure roulette lotteries $\mathbf{R}$ and $\mathbf{R}^*$ by First-order Stochastic Dominance Preference (via step 1) $\mathbf{R} \succ \mathbf{R}^*$ if and only if $V(\mathbf{R}) > V(\mathbf{R}^*)$. If there exists neither a best nor a worst outcome then starting with two outcomes x and y, with $\delta_x \succ \delta_y$, the above method can be used to construct $V(.)$ on the set $\{\mathbf{R} : \delta_x \succsim \mathbf{R} \succsim \delta_y\}$. For the extension of this representation to the rest of $\mathcal{L}$, the reader is referred to Step 5 in the proof of Theorem 2 in Machina and Schmeidler [7, Step 5, Proof of Theorem 5, p. 775].

Acknowledgments

We thank Edi Karni, Mark Machina and two anonymous referees for their useful comments and suggestions.

References

[1] F.J. Anscombe, R.J. Aumann, A definition of subjective probability, Ann. Math. Statist. 34 (1963) 199–205.
[2] P.C. Fishburn, Utility Theory for Decision Making, Wiley, New York, 1970. Reprinted Robert E. Kreiger Publishing Company, Huntington, NY, 1979.
[3] I. Gilboa, D. Schmeidler, Maxmin expected utility with a non-unique prior, J. Math. Econ. 18 (1989) 141–153.
[4] P. Ghirardato, M. Marinacci, Risk ambiguity and the separation of utility and beliefs, Math. Operations Res. 26 (2001), 864–890.
[5] E. Karni, A new approach to modeling decision making under uncertainty and defining subjective probabilities, mimeo, Johns Hopkins University, 2004.
[6] M. Machina, Almost-objective uncertainty, Econ. Theory 24 (2004) 1–54.
[7] M. Machina, D. Schmeidler, A more robust definition of subjective probability, Econometrica 60 (1992) 745–780.
[8] M. Machina, D. Schmeidler, Bayes without Bernoulli, J. Econ. Theory 67 (1995) 106–128.
[9] L.J. Savage, The Foundations of Statistics, Wiley, New York, 1954. Revised and enlarged, Dover Publications, New York, 1972.
[10] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (1989) 571–587.