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Bargaining and Boldness

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We study a multiperson bargaining problem with general risk preferences through the use of Shaked's game of cycling offers with exogenous breakdown. If preferences are "smooth," then as the risk of breakdown vanishes, the limiting outcome is one in which bargainers are *equally marginally bold*; where a bargainer's marginal boldness measures his willingness to risk disagreement in return for a marginal improvement in his position. Under smoothness, any (ordinal-)Nash solution is an equally marginally bold outcome. However, unlike the concept of the (ordinal-)Nash solution, a unique equally marginally bold outcome exists in natural cases—in particular, if all bargainers have risk-averse preferences of the rank-dependent expected utility type. For these preferences, the equally marginally bold outcome maximizes a "bargaining power"-adjusted (asymmetric) Nash product where the degree of asymmetry is determined by the disparity in the marginal valuation of certainty among bargainers. *Journal of Economic Literature* Classification Numbers: C72, C78, D81.
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1. INTRODUCTION

Bargainer's attitudes toward risk have long played a central role in many formal theories of bargaining. The notion that bargainers bear in mind the risk of a breakdown in negotiations has been recognized at least since Zeuthen (1930), but it was Nash's (1950, 1953) approach which caught on and held sway in economic modeling. However, expected utility theory, upon which Nash's analysis is built, has been challenged on introspective, experimental, and empirical grounds.[1] In response, several papers have recently examined generalizations of expected utility in the context of bargaining.[2] These works have been based, to some extent, on the use of suitable adaptations of Nash's orginal axioms for the bilateral bargaining problem. Although such axiomatizations are appealing, we doubt that the Nash solution would have made such inroads into economics simply on the strength of Nash's original abstract model and the axioms that he postulated. The overwhelming preference expressed by economists for the Nash solution in bargaining applications over other possible solutions—like the Kalai–Smorodinsky solution or the egalitarian solution—arises, we contend, from the support offered by more detailed strategic models in what we dub the Sutton program (1986), which calls (in the bilateral case) for the use of the alternating-offers game as a testing ground for bargaining axioms.

Our starting point in this paper is to note that the Sutton program may be used not to simply support an appealing set of axioms, but also to explore possible extensions of the Nash solution to a more general family of risk preferences, and to do that in a multiperson or multilateral framework. To carry out this extension, we follow a simple generalization of the alternating-offers protocol suggested by Shaked (a textbook reference is Osborne and Rubinstein, 1990 p. 63). In Shaked's game, bargainers rotate in making proposals, and the requirement for agreement is unanimity. If an offer is rejected, then there is a risk of breakdown of negotiations, yielding the disagreement outcome.

The connection between the interactive (or purely game-theoretic) and the decision-theoretic sides of this model is in itself worthy of some discussion and is a major contribution of this paper. In essence, any variation of the alternating-offers protocol is a story of temporal monopoly; an agreement is seen as the outcome of a dynamic game in which individuals

[1]See, for instance, Machina (1987) and the references therein.

[2]The seminal reference is Rubinstein, Safra, and Thomson (1992), which in turn has led to follow-up papers by Grant and Kajji (1994, 1995), Valenciano and Zarzuelo (1994), Hanany and Safra (2000) and Houba, Tieman, and Brinksma (1998). A somewhat different approach is taken in Safra and Zilcha (1993).

are each given in turn the right to ask their opponents to accept a sure payoff, or bear the risk of ending up with nothing. This choice problem is also the basis for a class of systematic violations of expected utility theory known as the certainty effect, or common ratio effect.[3] The fact that individuals "overvalue" outcomes that are considered certain relative to outcomes that are merely probable was first recognized by Allais (1953) and is perhaps the most systematically observed violation of the expected utility theory. Exploiting this relationship, we are able to (i) characterize stationary subgame-perfect equilibrium offers in Shaked's game for Allais-type preferences, and (ii) show that as the risk of breakdown vanishes, the limit of stationary subgame offers is an outcome in which all bargainers are *equally marginally bold*, where a bargainer's marginal boldness measures his willingness to risk disagreement in return for a marginal improvement in his position. This result offers a linkage between the degree of departure from expected utility and the outcome of bargaining. Indeed it is bargainers' attitude toward small risks—in particular, a measure of "probabilistic" risk aversion—which plays the major role in determining their bargaining power.

There are, of course, prices to pay. In this paper, bargainers have general preferences over simple lotteries, and compound lotteries are reduced by multiplying through the probabilities. Thus, at any stage of the rotating-offers game, bargainers care only about the distribution over outcomes induced by every possible continuation of the game, and not about the timing of resolution per se. In particular, our assumptions are incompatible with Machina's (1989) formulation of dynamic behavior for nonexpected utility decision makers, in which ex post preferences are derived from ex ante preferences by conditioning on any borne risk. They are also incompatible with Segal's (1990) approach, in which preferences are time separable but compound lotteries are not ranked by their induced distribution over final outcomes. In contrast, Karni and Safra's (1989) "behaviorally consistent" strategy for dealing with dynamic choice does fit into our framework. We find it reassuring that Karni and Safra's theory agrees with recent empirical findings on dynamic versions of common ratio problems that reject the approaches by Machina and Segal (see Cubitt, Starmer, and Sugden, 1998).

Another price is the differentiability assumption. As the name suggests, equally marginal boldness is a tangency condition that is not well defined without the differentiability of the function representing the bargainers' risk preferences over the class of binary lotteries obtained by mixing the disagreement outcome with any other single outcome. We contend, however,

[3]See Camerer (1995) for a review of such findings.

that the gain from the differentiability assumption far exceeds the cost, since practically all specific nonexpected utility functionals that have been used in theoretical and empirical research are differentiable in this sense.

The plan of this paper is as follows: After we set up the model in Section 2, in Section 3 we characterize the set of stationary subgame-perfect equilibrium in Shaked's game. We devote Section 4 to an exploration of the concept of marginal boldness. In Section 5 we present the convergence result. Finally, in Section 6 we explore the relation among equally marginally bold outcomes and generalizations to the multiperson case of the outcomes proposed by Rubinstein, Safra, and Thomson (1992) and by Safra and Zilcha (1993).

2. BASIC SETUP

There is a perfectly divisible object of size 1. There are I, $I \geq 2$, bargainers who negotiate over how the object should be divided. Denote by $x^i \geq 0$ the share that bargainer i receives. The set of feasible divisions, or the set of outcomes, is denoted by $\mathbf{X}$, with its generic element $\mathbf{x} = (x^1, \ldots, x^I)$; that is, $\mathbf{X} = \{\mathbf{x} : \forall i, x^i \geq 0, \sum_{i=1}^{I} x^i = 1\}$. Denote by Δ the interior of $\mathbf{X}$.

There is a given designated disagreement outcome, denoted by D. The set $\mathbf{X} \cup \{D\}$ is endowed with the natural σ-field, where the point D is regarded as a discrete point. Denote by $\mathscr{L}$ the set of lotteries (probability measures) with finite support over $\mathbf{X} \cup \{D\}$. An *elementary* lottery is one whose support consists of at most one outcome from $\mathbf{X}$ and the disagreement outcome D. We denote by $p\mathbf{x}$ the elementary lottery that yields the outcome $\mathbf{x}$ with probability p. The set of elementary lotteries is denoted by $\mathscr{L}_e$.

Bargainer i has a continuous preference relation $\succeq^i$ on $\mathscr{L}$. Throughout the paper, we keep the set of feasible divisions $\mathbf{X}$ and the disagreement outcome D fixed, so a bargaining problem is naturally identified with a collection of preferences $(\succeq^i)_{i=1}^{i=I}$. We assume that the object is "private" (in the sense that bargainer i is concerned only with his share of the pie) and "desirable" (so that preferences $\succeq^i$ are monotone with respect to the relation of first-order stochastic dominance, with strict dominance on $\mathscr{L}_e$.) We also assume that each bargainer views D as the least desirable. Thus, from bargainer's i point of view, D may be identified with (any) $\mathbf{x}$ in $\mathbf{X}$ for which $x^i = 0$.[4] Since the object is private, we represent agent i's preferences over $\mathscr{L}_e$ by a continuous function $V^i : [0,1] \times [0,1] \to \mathbb{R}$ given by the rule

[4]There is an analogy between our bargaining setting and a pure exchange economy with a public good, p, and a private good, x. Any elementary lottery $p\mathbf{x}$ induces marginal distributions $(p, x^1), \ldots, (p, x^I)$ that can be interpreted as bargainers' consumption bundles derived

$V^i(p, x) \geq V^i(q, y)$ if and only if $p\mathbf{x} \succeq^i q\mathbf{y}$ for any $\mathbf{x}, \mathbf{y}$ with $x^i = x$, $y^i = y$. We normalize V^i so that $V^i(p, 0) = 0$ for all p and $V^i(1, 1) = 1$. Notice that such V^i is unique up to monotonic transformation.

Later in the paper we assume that we can pick a function V^i as above for each i satisfying:

DUEL (differentiable utility over elementary lotteries): The restriction of $\succeq^i$ to $\mathscr{L}_e$ has a twice continuously differentiable representation V^i satisfying (i) $V^i_p(p, 0) = 0$ for all p in $[0, 1]$ and $V^i_p(p, x) > 0$ for all p in $[0, 1]$ and all x^i in $(0, 1]$, and (ii) $V^i_x(0, x) = 0$ for all x in $[0, 1]$, and $V^i_x(p, x) > 0$ for all p in $(0, 1]$ and all x in $[0, 1]$.[5]

Notice that condition DUEL is weak enough to accommodate preferences of the weighted utility class introduced by Chew (1983), of the class which Gul's (1991) dubbed "disappointment averse" (DA), and also preferences of the "rank-dependent" family, including Quiggin's (1982) "anticipated" or rank-dependent expected utility (RDEU) preferences, and Puppe's (1991) homogeneous RDEU preferences with prize-dependent distortion of the probabilities. Preference relations that allow a multiplicatively separable form, i.e., $V^i(p, x) = g^i(p)u^i(x)$, represent a natural reference point in our analysis. When g^i is linear, preferences are said to be disagreement linear (DL), which includes the expected utility (EU) preferences (see Grant and Kajii, 1995). Notice that preferences which do not have a multiplicatively separable representation on $\mathscr{L}$ may have one on $\mathscr{L}_e$. For instance, the DA functional has a multiplicatively separable representation. In this case, $g^i(p)$ has the form $p/[1 + (1 - p)\beta^i]$, where $\beta^i > -1$.

3. THE BARGAINING GAME

3.1. *Shaked's Rotating Offers Protocol*

Shaked's game is as follows. In each period there is a designated bargainer i who submits a proposal $\mathbf{x}_i = (x^1_i, \ldots, x^I_i) \in X$. The other bargainers $i + 1, \ldots, i + I - 1$ (modulus I) either accept or reject the proposal in this order. If all of them accept $\mathbf{x}_i$, then the game concludes, and each bargainer j receives x^j_i. If $\mathbf{x}_i$ is rejected by even just one bargainer, then with probability ρ, where $0 < \rho < 1$ is a fixed parameter, play passes to the next period, where bargainer $i + 1$ becomes the new proposer and bargainers $i + 2, \ldots, i + I$ in turn respond; and with probability $1 - \rho$, the

from $p\mathbf{x}$. Feasibility requires $1 \geq p \geq 0$, $x^i \geq 0$, and $\sum_{i=1}^{I} x^i = 1$. Efficiency implies that $p = 1$ (i.e., only degenerate lotteries are efficient).

[5] V^i_x is the partial derivative of V^i with respect to x. Similarly, V^i_p, V^i_{px}, and so on.

game concludes, and bargainers receive the disagreement outcome D. We assume that bargainer 1 proposes in the first period.

A *history* is a specification of a point in the game tree and the actions (proposals and rejections) taken at each decision node in the game up to that point. A (pure) *strategy* for bargainer i, denoted $\sigma^i = (\sigma^i_t)_{t=1}^{\infty}$, specifies a feasible action at every history at which he must act. A strategy profile is a I-tuple of strategies, one strategy for each bargainer.

Any continuation strategy profile $\sigma = (\sigma^i, \sigma^{-i})$ uniquely determines the agreed division $\mathbf{x} = (x^1, \ldots, x^I) \in \mathbf{X}$, and the period τ in which the agreement is reached or no agreement is ever reached. We assume that compound lotteries, in which the uncertainty is solved sequentially rather than simultaneously, are reduced to probability measures on $\mathscr{L}$ by the usual multiplication rule. Thus from bargainer i's viewpoint, the continuation strategy profile σ can be naturally identified as an elementary lottery $\rho^\tau x^i$ on $\mathscr{L}_e$, and the *payoff* of bargainer i is given by $V^i(\rho^\tau, x^i)$.

A strategy profile is *subgame perfect* if, at every history, it is a best response to itself. We say that σ is a *stationary subgame-perfect equilibrium* (SSPE) if σ subgame-perfect equilibrium and the actions that it prescribes in every period do not depend on time or on the actions in previous periods. In particular, each i's proposal is time independent, and thus we write $\mathbf{x}_i = (x_i^1, \ldots, x_i^I)$ for the proposal that bargainer i makes when he is the proposer. We refer to the $\mathbf{x}_i$, $i = 1, \ldots, I$, as SSPE proposals.

As Shaked showed in the expected utility case when $I > 2$, any feasible division may be supported using subgame-perfect pure strategies provided that ρ is large enough. As the same problem arises in our case where we are expanding the class of preferences for bargainers, we concentrate on SSPE from now on.[6]

3.2. *The Certainty Effect and SSPE Outcomes*

Within the EU framework, the characterization of SSPE outcomes in Shaked's game builds on the property that in any round the proposer can extract from the other bargainers any surplus in excess of their certainty equivalent from delaying agreement until the next round. In our setting, this means that bargainer i proposes $\mathbf{x}_i$ such that any other bargainer j gets a share x_i^j such that $V^j(1, x_i^j) = V^j(\rho, x_{i+1}^j)$ and all j accept. But if V is not linear in ρ and there are more than two bargainers, then this property is not sufficient for $\mathbf{x}_i$ to be a SSPE. To see this, consider a three-bargainers situation in which $V^3(1, x_1^3) = V^3(\rho, x_2^3)$ and $V^3(1, x_2^3) = V^3(\rho, x_3^3)$. Thus, in period 1, bargainer 3 is indifferent between accepting x_1^3 or rejecting it

[6]Additional support for the restriction to these kinds of strategies comes (at least in the two-person case) from the experimental work of Zwick, Rapoport, and Howard (1992).

and accepting x_2^3 in period 2. If the game reaches the second period, he is also indifferent between accepting x_2^3 or rejecting it and obtaining x_3^3 in period 3. However, if bargainer's 3 preferences are not linear in the probabilities, then it is possible that $V^3(1, x_1^3) < V^3(\rho^2, x_3^3)$, and therefore it is optimal for bargainer 3 to reject the proposals of bargainers 1 and 2 until it is his turn and then obtain x_3^3 for himself (conditional on the game not terminating). This would be the case if bargainer 3, who is willing to insure himself by accepting a sure x_2^3 in exchange for a ρ chance of receiving x_3^3 and otherwise nothing, would be willing to do so for a less favorable outcome if more uncertainty existed. This particular violation of the independence axiom, called the certainty effect, is one of the best-known findings of the experimental literature, and therefore what could be considered an exception is in fact the rule in a non-EU analysis. Keeping this in mind, we take the route of concentrating on preferences accommodating this effect.[7] Formally, we assume that each bargainer i's preferences satisfy:

DICE (disagreement increasing certainty equivalence): If $\mathbf{y} \sim^i q\mathbf{x}$, then $r\mathbf{y} \preceq^i (rq)\mathbf{x}$ for any probability r.

This condition states that bargainers are no more risk averse when there is a chance of getting the worst outcome than where there is no such chance. Empirical support for DICE is extensive, since it excludes only preferences violating *both* expected utility theory and the certainty effect. Notice that DICE is implied by (i) the *weak homogeneity* assumption in Grant and Kajii (1995) that characterizes DL preferences, (ii) the *subproportionality* property in Kahneman and Tversky's (1979) "prospect theory," and (iii) Machina's hypothesis II, or "family out" (see Fig. 1). For the class of preferences with a multiplicatively separable representation $V^i(p, x) = g^i(p)u^i(x)$, DICE is satisfied if and only if for all probabilities p and q, $g^i(p)g^i(q) \leq g^i(pq)$; that is, if and only if the elasticity of g^i is increasing.[8] Thus, in particular, all DA preferences satisfy DICE.

The following lemma demonstrates that if each bargainer's preference relation satisfies DICE, then we can succinctly characterize the set of SSPE proposals by proving that in any SSPE equilibrium, i will offer j the certainty equivalent of the amount that j would get in the next period in which he is the proposer if all intermediate offers are rejected. DICE guarantees that if bargainer j is willing to reject today's offer by i to accept tomorrow's

[7] At the small cost of precluding violations of EU in the *opposite* direction to the one predicted by the certainty effect, this restriction allows for a sharp characterization of SSPE as the fixed points of an smooth operator using certainty equivalents (Lemma 3.1), which generally cannot be done when $I > 2$.

[8] Lemma 4.1 in Segal (1987, p. 195).

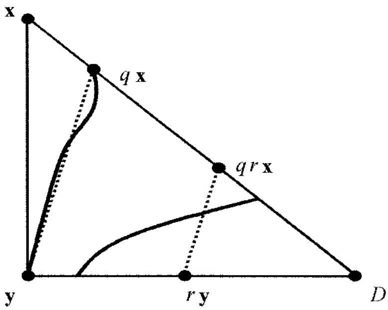

FIG. 1. DICE and the "fanning out" hypothesis in the Marschak–Machina triangle.

offer by $i+1$, then he must be also willing to reject $i+1$'s offer in anticipation of accepting the next period's proposal by $i+2$. The argument applies iteratively until it is j's turn to make a proposal. As a conclusion, i's optimal offer to j depends only on what j will propose for himself.

Let $\tau(i,j)$ be $j-i$ if $i \le j$ and $I-(i-j)$ otherwise (i.e., if i is the proposer in period t, then j is the proposer in period $t+\tau(i,j)$). Then

LEMMA 3.1. *Suppose that DICE holds. The proposals $\mathbf{x}_i$, $i = 1, \ldots, I$, corresponding to a SSPE are always accepted, and they are characterized by*

$$V^j(1, x_i^j) = V^j\left(\rho^{\tau(i,j)}, x_j^j\right) \qquad \textit{for all } i, j. \tag{1}$$

Proof. We proceed in four stages:

1. *There is no SSPE where all proposals are rejected.* By contradiction, suppose that there was a SSPE where an agreement is never reached; in particular, bargainer 1's proposal is rejected in period 1. Consider the (possibly) off-equilibrium subgame that commences in period 1 after bargainer 1 has made an equal share proposal, i.e., $x_1^j = 1/I$ for all $j \in \{1, \ldots, I\}$. Since strategies are stationary, this subgame can be analyzed in reduced form as a sequential move game where, in order, each bargainer $j = 2, \ldots, I$ decides whether to accept or reject the proposal, and such that any rejection leads to the disagreement outcome with probability 1. A straightforward backward-induction argument shows that the subgame-perfect equilibrium decision for each bargainer in this reduced single round of the bargaining game is to accept the proposal. Hence an equal share proposal for bargainer 1 in period 1 strictly dominates his proposal in the putative SSPE, which, by assumption, is rejected and leads in equilibrium to the disagreement outcome with probability 1. Therefore, no SSPE exists in which offers are rejected in every round.

2. *In a SSPE, proposals are always accepted.* If not, there is a bargainer i whose proposal is rejected, and bargainer $i+1$'s proposal $\mathbf{x}_{i+1}$ is accepted in

the SSPE, since in the previous stage we have shown that some bargainer's proposal must be agreeable. Since utility is increasing in p, there must be a proposal $\mathbf{x}_i$ such that $V^j(1, x_i^j) > V^j(\rho, x_{i+1}^j)$ for each j. But if such an $\mathbf{x}_i$ is proposed by bargainer i, then it must be accepted, since if it is rejected, $\mathbf{x}_{i+1}$ will be the final outcome because of the stationarity. Then bargainer i would be better off, which is a contradiction.

3. *In a SSPE, proposals satisfy Eq.* (1): Let $\mathbf{x}_i$ be a SSPE outcome in the period where bargainer i is the proposer, $i = 1, \ldots, I$. Fix a bargainer i. Then for any $j, j \neq i, V^j(1, x_i^j) < V^j(\rho^{\tau(i,j)}, x_j^j)$ cannot be the case, since otherwise bargainer j rejects proposals until it is his turn and then proposes $\mathbf{x}_j$, and $\mathbf{x}_j$ is accepted. This is equivalent to bargainer j receiving x_j^j with probability $\rho^{\tau(i,j)}$ and D with probability $1 - \rho^{\tau(i,j)}$ at the time when i makes a proposal, thus accepting x_i^j is not a best response for j. Hence $V^j(1, x_i^j) \geq V^j(\rho^{\tau(i,j)}, x_j^j)$ must hold for any i and j. Now we show that $V^j(1, x_i^j) = V^j(\rho^{\tau(i,j)}, x_j^j)$ for any $i \neq j$, by induction on $n = \tau(i, j)$. First, pick any i and j with $\tau(i, j) = 1$ (i.e., j is next to i). Suppose that $V^j(1, x_i^j) > V^j(\rho, x_j^j)$ holds. Then in the subgame where bargainer i proposes y_i^j such that $V^j(1, x_i^j) > V^j(1, y_i^j) > V^j(\rho, x_j^j)$, bargainer j must accept the proposal. Thus bargainer i is not optimizing, a contradiction. This establishes the claim for $n = 1$. Now let $n > 1$ and suppose that $V^j(1, x_i^j) = V^j(\rho^{\tau(i,j)}, x_j^j)$ holds for any $i \neq j$ with $\tau(i, j) < n$. Pick any i and j with $\tau(i, j) = n$. For any k such that $\tau(k, j) = 1, \ldots, n-1, V^j(1, x_k^j) = V^j(\rho^{\tau(k,j)}, x_j^j)$ holds by the induction hypothesis. Thus by DICE, we have $V^j(\rho^{n-\tau(k,j)}, x_k^j) \leq V^j(\rho^n, x_j^j)$ for any k such that $\tau(k, j) = 1, \ldots, n-1$. This shows that in any subgame where $j = i + n$ has rejected bargainer i's offer, his (ex ante) payoff is at most $V^j(\rho^n, x_j^j)$. Thus if bargainer i can offer y_i^j with $V^j(1, x_i^j) > V^j(1, y_i^j) > V^j(\rho^n, x_j^j)$, then it must be accepted. Therefore, $V^j(1, x_i^j) \leq V^j(\rho^n, x_j^j)$, which together with $V^j(1, x_i^j) \geq V^j(\rho^n, x_j^j)$ implies $V^j(1, x_i^j) = V^j(\rho^n, x_j^j)$, as desired.

4. *Proposals* $\mathbf{x}_i$, $i = 1, \ldots, I$, *satisfying Eq.* (1) *can be supported in a SSPE.* Pick any proposals $\mathbf{x}_i$, $i = 1, \ldots, I$, satisfying Eq. (1). Consider the stationary strategy for bargainer i where if it is his turn to be the proposer, he proposes $\mathbf{x}_i$, and if some other individual $j \neq i$ is the proposer, i accepts a proposal $\mathbf{x}_j$ whenever $V^i(1, x_j^i) \geq V^i(\rho^{\tau(j,i)}, x_i^i)$ and rejects it otherwise. We show that this profile of strategy constitutes an SSPE. Since we can truncate the game to be of finite length and bargainers still receive the same payoff, if a stationary strategy profile is not a subgame-perfect equilibrium, then there exists a bargainer with a profitable one-shot deviation (see Lemma 98.2 in Osborne and Rubinstein, 1994). Hence it suffices to

show that there is no profitable one-shot deviation for each bargainer i. We split this proof into two cases, depending on the role of proposer or responder of bargainer i:

- Bargainer i is the proposer in period t. If i proposes more than x_i^i for himself, then there exists at least one bargainer j who is now offered $\hat{x}_i^j < x_i^j$. Since $V^j(1, \hat{x}_i^j) < V^j(\rho^{\tau(i_t, j)}, x_j^j)$, j will reject. So by proposing more, i will receive x_{i+1}^i with probability ρ and D with probability $1 - \rho$. But we have $V^i(1, x_{i+1}^i) = V^i(\rho^{I-1}, x_i^i) < V^i(1, x_i^i)$, and so $V^i(\rho, x_{i+1}^i) < V^i(1, x_i^i)$ holds, because V^i is increasing in p. Hence i's proposal is optimal.
- Bargainer $j \neq 1$ is the proposer. Bargainer i is offered x_j^i, where $V^i(1, x_j^i) = V^i(\rho^{\tau(j,i)}, x_i^i)$. If he rejects, then he will receive x_{j+1}^i with probability ρ and D with probability $1 - \rho$, where $V^i(1, x_{j+1}^i) = V^i(\rho^{\tau(j,i)-1}, x_i^i)$. And by DICE, $V^i(\rho, x_{j+1}^i) \leq V^i(\rho^{\tau(j,i)}, x_i^i) = V^i(1, x_j^i)$. So i's acceptance rule is optimal. ■

Thus, provided that a solution $\mathbf{x}_i$, $i = 1, \ldots, I$, to Eq. (1) exists and preferences satisfy DICE, the equilibrium path of a SSPE consists of the first bargainer proposing $\mathbf{x}_1$ and everyone accepting it (which ends the game). Hence, $\mathbf{x}_1$ is the outcome induced by such SSPE, or simply a SSPE outcome.

REMARK: In the proof of Lemma 3.1, we use DICE *only* when $I > 2$. Thus, for the two-person case, Lemma 3.1 derives from first-order stochastic dominance alone. Notice that in this case, SSPE proposals are characterized by Rubinstein-like equations $V^1(1, x_2^1) = V^1(\rho, x_1^1)$ and $V^2(1, x_1^2) = V^2(\rho, x_2^2)$.

3.3. *Existence and Uniqueness of SSPE*

We first show that under DICE,[9] an SSPE outcome exists and every SSPE outcome is better than D for every bargainer. To do this, we write $\phi(\rho; z^1, \ldots, z^I) = [1 - \sum_{j \neq i} c^j(\rho^{\tau(i,j)}, z^j)]_{i=1}^I$, where $c^i : [0, 1] \times [0, 1] \to [0, 1]$ denotes the certainty equivalent function for bargainer i, given implicitly by $V^i(p, x) = V^i(1, c^i)$. By Lemma 3.1, the problem of finding an SSPE outcome under DICE is equivalent to finding a solution to Eq. (1), which is then equivalent to finding $(z^1, \ldots, z^I)$ in $[0, 1]^I$ such that $\phi(\rho; z^1, \ldots, z^I) = (z^1, \ldots, z^I)$.

PROPOSITION 3.1. *If preferences satisfy DICE, then there exists an SSPE outcome, and every SSPE outcome is strictly positive.*

[9] Again, from a formal standpoint, this statement is unnecessarily strong. DICE matters in this paper only as a sufficient condition for the characterization of Lemma 3.1 to hold. In the two-person case, this result is true even without DICE, and so are the remaining statements concerning existence, uniqueness, and convergence of SSPE outcomes.

Proof. Let $\tilde{\phi}(\rho; z^1, \ldots, z^I) = [\max\{0, 1 - \sum_{j\neq i} c^j(\rho^{\tau(i,j)}, z^j)\}]_{i=1}^I$. Clearly, any fixed point of $\phi(\rho; \cdot)$ is a fixed point of $\tilde{\phi}(\rho; \cdot)$ and $\phi(\rho; \cdot) = \tilde{\phi}(\rho; \cdot)$ for strictly positive z. So we are done if $\tilde{\phi}(\rho; \cdot)$ has a fixed point and every fixed point of $\tilde{\phi}(\rho; \cdot)$ is strictly positive. The existence of a fixed point follows from Brouwer's theorem, since $\tilde{\phi}(\rho; \cdot)$ is a continuous function from the compact set $[0,1]^I$ into itself. To show that any fixed point is strictly positive, pick any fixed point $\bar{z}$ of $\tilde{\phi}(\rho; \cdot)$ and suppose that on the contrary $\bar{z}$ is not strictly positive. Since $\tilde{\phi}(\rho; 0, \ldots, 0) = (1, \ldots, 1)$, at least one $\bar{z}^i$ must be positive, so, without loss of generality, assume that $\bar{z}^1 = 0$ and $\bar{z}^2 > 0$. Since $\bar{z}$ is a fixed point of $\tilde{\phi}$, we have $\bar{z}^2 = 1 - \sum_{j\neq 2} c^j(\rho^{\tau(2,j)}, \bar{z}^j)$, and $\bar{z}^1 = 0$ implies $1 \leq \sum_{j=2}^I c^j(\rho^{\tau(1,j)}, \bar{z}^j)$. But from $\rho^{\tau(1,j)} < \rho^{\tau(2,j)}$ for $j \neq 1, 2$, and since the certainty equivalent function c^j is increasing in probability, we have

$$\begin{aligned}
1 &\leq \sum_{j=2}^{I} c^j\left(\rho^{\tau(1,j)}, \bar{z}^j\right) = c^2\left(\rho, \bar{z}^2\right) + \sum_{j=3}^{I} c^j\left(\rho^{\tau(1,j)}, \bar{z}^j\right) \\
&\leq c^2\left(\rho, \bar{z}^2\right) + \sum_{j=3}^{I} c^j\left(\rho^{\tau(2,j)}, \bar{z}^j\right) \\
&= c^2\left(\rho, \bar{z}^2\right) + \left(\sum_{j\neq 2} c^j\left(\rho^{\tau(2,j)}, \bar{z}^j\right) - c^1\left(\rho^{\tau(2,1)}, \bar{z}^1\right)\right) \\
&= c^2\left(\rho, \bar{z}^2\right) + 1 - \bar{z}^2 - c^1\left(\rho^{\tau(2,1)}, \bar{z}^1\right).
\end{aligned}$$

Since $c^1(\rho^{\tau(2,1)}, \bar{z}^1) = 0$, the foregoing calculation shows that $1 \leq c^2(\rho, \bar{z}^2) + 1 - \bar{z}^2$, and hence $\bar{z}^2 \leq c^2(\rho, \bar{z}^2)$. This contradicts the fact that $c^2(\rho, \bar{z}^2)$ is the certainty equivalent of elementary lottery $\rho\bar{z}^2$. ■

One obvious approach to establishing sufficient conditions for the uniqueness of equilibrium is to find conditions under which ϕ has a unique fixed point. In the EU case, Merlo and Wilson (1995) provide the following sufficient condition on each bargainer's preferences, which holds whenever the von Neumann–Morgenstern utility function of such bargainer is concave:

CEC (certainty equivalent contraction): $c^i(p, \cdot)$ is a contraction for any $p < 1$.[10]

Notice that, together with DUEL, CEC implies:

CEC* $c_x^i(p, \cdot) < 1$ for any $p < 1$.

[10]That is, given $p < 1$, there is a $\delta < 1$ such that for any $x, y \in [0,1]$, $|c^i(p,x) - c^i(p,y)| < \delta|x - y|$.

In our more general framework, however, CEC* alone does not guarantee uniqueness. An additional potential source of multiplicity is the existence of nonmonotonic cross-effects between probabilities and prizes. Thus, we require:

IMCE (increasing marginal certainty equivalent): $c^i_{xp}(p, x) > 0$ for all $p \in (0, 1]$ and all $x \in (0, 1]$.

Whereas CEC* implies that the (absolute) risk premium for an elementary lottery, $x - c(p, x)$, is an increasing function of x, IMCE implies that the rate of this increase is increasing in the likelihood that the elementary lottery results in the disagreement outcome. Under DUEL, conditions CEC* and IMCE both hold whenever $V^i_{xx}(1, x) \leq 0$ and $V^i_{xp}(p, x) > 0$ for all $x \in (0, 1]$ and all $p \in (0, 1]$.[11] To see this, notice that $V^i_{xp}(p, x) > 0$ implies that $V^i_x(p, x) < V^i_x(1, x)$, and $V^i_{xx}(1, x) \leq 0$ ensures that $V^i_x(1, x) < V^i_x(1, c^i(p, x))$, since $c^i(p, x) < x$ by construction. Thus, $c^i_x(p, x) = \frac{V^i_x(p,x)}{V^i_x(1,c^i)} < \frac{V^i_x(1,x)}{V^i_x(1,c^i)} \leq 1$. Differentiating with respect to p and nothing that c^i is increasing in both arguments, we get $c^i_{xp}(p, x) = [V^i_{xp}(p, x) - V^i_{xx}(1, c^i)c^i_p(p, x)c^i_x(p, x)]/(V^i_x(1, c^i)) > 0$.

PROPOSITION 3.2. *Under DICE and DUEL, if preferences satisfy CEC* and IMCE, then the SSPE outcome is unique for any* $\rho \in (0, 1)$ *and is a continuously differentiable function of* ρ *on* $\rho \in (0, 1)$.

Proof. It suffices to show that ϕ has a unique fixed point or, equivalently, for any fixed $\rho \in (0, 1)$, the vector field $\Phi(\rho; z^1, \ldots, z^I) \equiv z - \phi(\rho; z^1, \ldots, z^I)$ defined on $(0, 1)^I$ has a unique zero. We first show that the Jacobian matrix of $\Phi(\rho; \cdot)$ has a positive determinant everywhere; i.e., the determinant of the matrix

$$\begin{pmatrix} 1 & c^2_x(\rho, z^2) & \cdots & \cdots & c^I_x(\rho^{I-1}, z^I) \\ c^1_x(\rho^{I-1}, z^1) & 1 & c^3_x(\rho, z^3) & \cdots & \vdots \\ \vdots & c^2_x(\rho^{I-1}, z^2) & 1 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ c^1_x(\rho, z^1) & c^2_x(\rho^2, z^2) & \cdots & c^{I-1}_x(\rho^{I-1}, z^{I-1}) & 1 \end{pmatrix} \quad (2)$$

is positive. Condition CEC* guarantees that (i) every off-diagonal entry is strictly less than 1, IMCE implies that (ii) for any $i \neq j$, the element (i, j)

[11]These conditions are satisfied in natural cases; concavity of the certainty utility function is prevalent in generalized risk preference models. For instance, in the case of RDEU preferences, it is a necessary condition for risk aversion. A positive cross-derivative also seems plausible, as it states, that as the marginal value of improving one's allocation increases, the more likely that there will be an agreement. In fact, for any multiplicatively separable functional, $V^i_{px}(p, x) > 0$ is implied by DUEL alone.

is smaller than the element $(i+1, j)$. Theorem A.1 in the Appendix shows that a square matrix with 1's in the diagonal and off-diagonal elements satisfying (i) and (ii) has positive determinant.

Then by the index theorem,[12] $\Phi(\rho; \cdot)$ has a unique zero, which can be written as a C^1 function by the implicit function theorem. ■

4. BOLDNESS

We shall show in the next section that as the risk of breakdown in Shaked's game converges to 0 (i.e., as $\rho \to 1$), the sequence of SSPE outcomes converges to an outcome in which one may view all of the bargainers as equally marginally bold. Marginal boldness is a "generalized utility" analog of Aumann and Kurz's (1977) boldness measure of attitudes toward risking large losses, in contrast with the Arrow–Pratt measures of risk aversion, which measure attitudes toward small risks only. Let us consider a given proposed division $\mathbf{x} = (x^1, \ldots, x^I) \in \Delta$ and imagine that bargainer i demands a small addition, $s > 0$, to his share. He will be prepared to risk a probability of disagreement up to the level $q(s)$, where $q(s)$ is implicitly defined by $V^i(1 - q(s), x^i + s) - V^i(1, x^i) = 0$. We define the marginal boldness of bargainer i at outcome $\mathbf{x}$ as $q'(0)$, i.e., the limit (when it exists) of the difference quotient $\frac{q}{s}$ as s tends to 0. By choosing V^i such that $V_p^i(1, x^i)$ and $V_p^i(1, x^i)$ are both positive if $x^i > 0$, this leads to:

DEFINITION 4.1. The marginal boldness $b^i(\mathbf{x})$ of bargainer i at $\mathbf{x} \in \mathbf{X}$ is defined as

$$b^i(\mathbf{x}) = \begin{cases} \frac{V_x^i(1, x^i)}{V_p^i(1, x^i)} & \text{for } x^i > 0 \\ +\infty & \text{for } x^i = 0. \end{cases} \tag{3}$$

Intuitively, the marginal boldness offers a first-order approximation to the cost for bargainer i of risking x^i (measured in terms of outcome). Clearly, the marginal boldness is invariant to any *ordinal* transformation of the function V^i; thus the concept of marginal boldness does not depend on the choice of utility function. Notice also that in the expected utility cases, $b^i(\mathbf{x}) = u_x^i(x^i)/u^i(x^i)$, which is the Aumann–Kurz marginal boldness measure.

Continuing the analogy with the Aumann–Kurz analysis, let us define:

DEFINITION 4.2. Outcome $\mathbf{x} \in \Delta$ is deemed *equally marginally bold* (EMB) for the bargainers if $b^i(x) = b^j(x)$ for all i, j.

[12] See, for instance, Milnor (1965).

Thus, at any EMB outcome the cost of risking their share is the same for all bargainers. An attractive feature of the EMB concept is that it exists under fairly weak assumptions on the utility representations. Namely,

PROPOSITION 4.1. *There exists an EMB outcome if for each bargainer* i, V^i *is* C^1, (*i*) $V_p^i(1,0) = 0$ *and* $V_p^i(1,x) > 0$ *for all* x *in* $(0,1]$, *and* (*ii*) $V_x^i(1,x) > 0$ *for all* x *in* $[0,1]$.

Proof. See the Appendix. ■

In particular, if condition DUEL is satisfied, then an EMB outcome exists. For multiplicatively separable preferences, it can be readily seen that the equally marginally bold outcomes are characterized by the *first-order conditions* of the following maximization program:

$$\mathbf{x} = \arg\max_{y \in \mathbf{X}} \prod_{i=1}^{I} \left[u^i(y^i)\right]^{1/g_p^i(1)}. \tag{4}$$

Notice the similarity with the "bargaining power"-adjusted (i.e., asymmetric) Nash bargaining outcome in the expected utility case. In the foregoing program, we can interpret $\theta^i = [g_p^i(1)]^{-1}/(\sum_{j=1}^{I}[g_p^j(1)]^{-1})$ as bargainer i's bargaining power. This is quite intuitive, as $g_p^i(1)$ may be viewed as the bargainer's marginal evaluation of certainty for a given prize (that is better than the disagreement outcome). The greater the $g_p^i(1)$, the larger the premium placed on certainty by bargainer i, and hence other things being equal, the *less* bold is the bargainer in pushing for a better split. If bargainer i is weakly risk averse and $g^i(1) = 1$, then it follows that $g_p^i(1) \geq 1$. Thus the relative "overvaluation" of certainty by the bargainer (i.e., the degree of "probabilistic" risk aversion of the bargainer) in conjuction with the degree to which his marginal utility for the pie diminishes determines the EMB outcome.

A simple sufficient condition for the uniqueness of the EMB outcome is that each bargainer's marginal boldness is negatively related to his share of the pie; i.e., if for all $\mathbf{x}, \mathbf{y}$ with $x^i \neq y^i$,

$$\left[b^i(\mathbf{x}) - b^i(\mathbf{y})\right](x^i - y^i) < 0. \tag{5}$$

Indeed, if $\mathbf{x}$ and $\mathbf{y}$ are equally marginally bold outcomes with $\mathbf{x} \neq \mathbf{y}$, then without loss of generality, then we can assume that $x^1 > y^1$ and $x^2 < y^2$. But this implies that $b^1(\mathbf{x}) < b^1(\mathbf{y})$ and $b^2(\mathbf{x}) > b^2(\mathbf{y})$ by Eq. (5), which contradicts the fact that both $\mathbf{x}$ and $\mathbf{y}$ are EMB outcomes.

It is straightforward to check that if b^i is strictly decreasing for all i, then Eq. (5) holds, and thus uniqueness of the EMB outcome follows. If condition DUEL is assumed, since $c_p^i(1,x) = V_p^i(1,x)/V_x^i(1,x)$, from Eq. (3) we have $c_p^i(1,x) = 1/b^i(\mathbf{x})$ for any $\mathbf{x}$ with $x^i = x$, and thus $c_{xp}^i(1,x) = c_{px}^i(1,x) = -b_x^i(\mathbf{x})$. Thus, a positive cross-derivative for c^i guarantees the existence of a unique EMB outcome. Hence,

PROPOSITION 4.2. *Given DUEL, if preferences satisfy IMCE, then the EMB outcome is unique.*

5. THE LIMIT RESULT

In this section we obtain a characterization of the limit of SSPE outcomes as the parameter $1-\rho$ measuring the risk of breakdown tends to 0. We assume DICE and DUEL throughout this section. So, by Proposition 3.2, we write the unique SSPE outcome as $\mathbf{x}(\rho)$ for $\rho \in (0,1)$, and there is a unique EMB outcome by Proposition 4.2. The next result shows that the SSPE outcome approaches to a unique EMB outcome as probability of breakdown goes to 0.

PROPOSITION 5.1. *Assume that DICE, DUEL, and IMCE hold. Then for each j, $\bar{x}^j \equiv \lim_{\rho\to 1} x_j^j(\rho)$ exists, and $\bar{\mathbf{x}} = (\bar{x}^1, \ldots, \bar{x}^I)$ is a unique EMB outcome.*

Proof. Pick any increasing sequence $\{\rho_n : n = 1, \ldots\}$ with $\lim_{n\to\infty} \rho_n = 1$. Since an EMB outcome is unique and $\mathbf{X}$ is compact, it suffices to show that $\bar{\mathbf{x}}$ is an EMB outcome, assuming $\mathbf{x}_i^i(\rho_n) \to \bar{\mathbf{x}}^i$ for each i as $\rho_n \to 1$.

First, we claim that for every i, if $\bar{x}^j > 0$, then

$$\lim_{\rho_n\to 1} \frac{x_j^j(\rho_n) - x_i^j(\rho_n)}{1-\rho_n^{\tau(i,j)}} = \frac{1}{b^j(\bar{\mathbf{x}})} \text{ for all } j \neq i, \tag{6}$$

where $1/b^j(\bar{\mathbf{x}}) = 0$ if $b^j(\bar{\mathbf{x}}) = +\infty$, by convention; see Eq. (3). To see this, first recall that $\mathbf{x}(\rho_n)$ satisfies Eq. (1) by Proposition 3.1 and $x_i^j(\rho_n) < x_j^j(\rho_n)$ for each $i, j, i \neq j$. Applying the mean value theorem to the function $t \mapsto V^j(t + (1-t)\rho^{\tau(i,j)}, t x_i^j(\rho) + (1-t) x_j^j(\rho))$ for every n and for every pair i and j $(i \neq j)$, we can find some $(q_i^j(\rho_n), s_i^j(\rho_n))$ with $q_i^j(\rho_n) \in [\rho_n^{\tau(i,j)}, 1]$ and $s_i^j(\rho_n) \in [x_i^j(\rho_n), x_j^j(\rho_n)]$ such that

$$\begin{aligned}0 &= V^j\big(1, x_i^j(\rho_n)\big) - V^j\big(\rho_n^{\tau(i,j)}, x_j^j(\rho_n)\big)\\ &= V_p^j\big(q_i^j(\rho_n), s_i^j(\rho_n)\big)\big(1-\rho_n^{\tau(i,j)}\big) + V_x^j\big(q_i^j(\rho_n), s_i^j(\rho_n)\big)\big(x_i^j(\rho_n) - x_j^j(\rho_n)\big).\end{aligned}$$

Since $V_x^j > 0$ by DUEL, we can rewrite the foregoing expression as

$$\frac{V_p^j\big(q_i^j(\rho_n), s_i^j(\rho_n)\big)}{V_x^j\big(q_i^j(\rho_n), s_i^j(\rho_n)\big)} = \frac{x_j^j(\rho_n) - x_i^j(\rho_n)}{1-\rho_n^{\tau(i,j)}},$$

for any given n. As $\rho_n \to 1$, for all i and j we have $x_i^j(\rho_n) \to \bar{\mathbf{x}}^j$ from Eq. (1), and so $s_i^j(\rho_n) \to \bar{\mathbf{x}}^j$. Hence, the left-hand side converges to $1/b^j(\bar{\mathbf{x}})$, as desired.

Set $r^j = \frac{1}{b^j(\bar{\mathbf{x}})}$ for each j. Fix any i and j with $i \neq j$, and consider functions $\rho \mapsto [x_j^j(\rho) - x_i^j(\rho)]$ and $\rho \to (1 - \rho^{\tau(i,j)})$ defined on $[\rho_n, 1]$ for any given n. By Cauchy's mean value theorem, there is $\rho_n^* \in [\rho_n, 1]$ with

$$\frac{\frac{d}{dp}\left[x_j^j(\rho_n^*) - x_i^j(\rho_n^*)\right]}{\frac{d}{dp}\left[(1 - \rho^{\tau(i,j)})\right]\Big|_{\rho=\rho_n^*}} = \frac{\left[x_j^j(\rho_n) - x_i^j(\rho_n)\right]}{\left(1 - \rho_n^{\tau(i,j)}\right)}.$$

We have shown that the right-hand side converges to r^j, and so we have

$$\lim_{\rho_n \to 1} \frac{d}{d\rho}\left[x_j^j(\rho_n) - x_i^j(\rho_n)\right] = -\tau(i, j) r^j. \tag{7}$$

Since the equilibrium offers sum up to 1 by construction, $\sum_{j\neq 1}[x_j^j(\rho) - x_1^j(\rho)] - \sum_{j\neq i}[x_j^j(\rho) - x_i^j(\rho)] = 0$ holds for any $i \neq 1$. Differentiating this relation with respect to ρ and using Eq. (7), we get $(\sum_{j\neq 1} -\tau(1, j)r^j) - (\sum_{j\neq 1} -\tau(i, j)r^j) = -\tau(1, i)r^i + \tau(i, 1)r^1 - \sum_{j\neq 1,\, i} (\tau(1, j) - \tau(i, j))r^j = 0$. Note that $\tau(1, j) - \tau(i, j) = i - 1$ if $j > 1$, and it is equal to $(I - i + 1)$ otherwise. So, for each $i = 2, \ldots, I$, we can rewrite this relation as

$$-(i-1)\alpha_i + (I - i + 1)(\bar{r} - \alpha_i) = 0, \tag{8}$$

where $\alpha_i = \sum_{j=1}^{I} r^j$ and $\bar{r} = \sum_{j=1}^{I} r^j$. Hence, $\alpha_i = \bar{r}(I - i + 1)/I$ for each $i = 2, \ldots, I$. Thus, $r^1 = \cdots = r^I$, as desired. ■

6. DISCUSSION

Let us conclude with a discussion on the connection between the concept of EMB outcome and the other extensions of the Nash solution to general risk preferences in the literature.

6.1. *Comparison with Rubinstein, Safra, and Thomson's Extension*

Rubinstein, Safra, and Thomson's *ordinal Nash solution* associates with a two-person bargaining problem $(\succeq^i)_{i=1}^{i=2}$ its ordinal Nash outcome (or simply Nash outcome), defined as the element $\mathbf{x} \in \mathbf{X}$ such that for both i there is no $p \in [0, 1]$ and $\mathbf{y} \in \mathbf{X}$ such that $p\mathbf{y} \succ^i \mathbf{x}$ and $p\mathbf{x} \prec^j \mathbf{y}$. In our setting, one may wish to interpret this outcome $\mathbf{x}$, as one against which neither bargainer can extract a concession from the other because of the common risk of disagreement, $1 - p$, that each perceives exists if he sticks with his position.

To compare the concept of ordinal Nash outcome with the concept of EMB outcome, one must first extend Rubinstein, Safra, and Thomson's definition to our multiperson setting. The critical issue here is to define how the cost of a concession sought by bargainer i can be allocated among the other bargainers. A natural extension for a multilateral bargaining setting is to allow a bargainer to seek concessions simultaneously from any number of the other bargainers. This is reinforced by the public good nature of the probability of disagreement. More specifically, if bargainer i threatens to leave the negotiation table, then the negative public effect from disagreement will be borne by *all* of the bargainers, considerably increasing bargainer i's ability to extract concessions. Unfortunately, although this approach is intuitively appealing, we show in the Appendix that if $I > 2$, then no outcome survives this criterion, even for smooth EU preferences.

One way to recover existence of Nash outcomes in a multiperson setting is to allow a bargainer to seek a concession from only *one* other bargainer. Notice that this smaller flexibility in "financing" a given concession makes it much more difficult for a bargainer to appeal successfully against a given allocation. More formally, this second scenario leads to the following:

DEFINITION 6.1. Bargainer i can appeal against $\mathbf{x}$ if there exists $j, q \in [0, 1]$ and $s > 0$, such that $x^j - s \geq 0, V^i(1 - q, x^i + s) > V^i(1, x^i)$ and $V^j(1 - q, x^j) < V^j(1, x^j - s)$. Here $\mathbf{x}$ is an ordinal Nash outcome if no bargainer can appeal against $\mathbf{x}$.

Notice that the ordinal Nash outcome is invariant to monotonic transformations of V^i. Obviously, this coincides with Rubinstein, Safra, and Thomson's solution when $I = 2$.

To establish the link between EMB outcomes and ordinal Nash outcomes, for any $\mathbf{x} \in \mathbf{X}$ and every i, let $\alpha^i_{\mathbf{x}} : (q, s) \longmapsto V^i(1 - q, x^i + s) - V^i(1, x^i)$ and $\beta^i_{\mathbf{x}} : (q, s) \longmapsto -V^i(1 - q, x^i) + V^i(1, x^i - s)$. Then it can be readily shown that bargainer i has no successful appeal against $\mathbf{x}$ if and only if for each $j \neq i, (q, s) = (0, 0)$ maximizes $\alpha^i_{\mathbf{x}}$ subject to $\beta^j_{\mathbf{x}} = 0$. The first-order necessary condition for this maximization problem is equivalent to $b^i(\mathbf{x}) = b^j(\mathbf{x})$. So we have:

PROPOSITION 6.1. *Any ordinal Nash outcome is an EMB outcome.*

Of course, the converse will hold if the first-order condition is sufficient:

PROPOSITION 6.2. *Suppose that* $\mathbf{x}$ *is an EMB outcome. If all* $\alpha^i_{\mathbf{x}}$ *and* $\beta^j_{\mathbf{x}}$ *are quasi-concave, then* $\mathbf{x}$ *is a Nash outcome.*[13]

[13] In the two-person case, Houba, Tieman, and Brinksma (1998) provide weaker sufficient conditions for the existence of an ordinal Nash outcome if preferences are multiplicatively separable. Hanany and Safra (2000) characterize a set of preferences with the properties that

Thus, in light of Proposition 5.1, we can conclude that if all $\alpha_{\mathbf{x}}^i$ and $\beta_{\mathbf{x}}^j$ are quasi-concave, then the concept of Nash outcome captures the logic behind the rotating-offers game. However, $\alpha_{\mathbf{x}}^i$ and $\beta_{\mathbf{x}}^j$ are not necessarily quasi-concave under our assumptions on preferences. Notice that, because of its additive structure, $\beta_{\mathbf{x}}^j$ is concave (and so quasi-concave) if $V^j(1-q, x^j)$ is convex in q and $V^i(1, x^j - s)$ is concave in s. In the multiplicatively separable case, these conditions follow if u^j is concave and g^j is convex, so that for the RDEU case, quasi-concavity of $\beta_{\mathbf{x}}^j$ follows from aversion to mean-preserving spreads.[14] Even in this case, however, $\alpha_{\mathbf{x}}^i$ can fail to be quasi-concave, and the Nash outcome (which would be unique, since aversion to mean-preserving spreads is satisfied) might not exist. A family of two-person examples of this situation can be constructed as follows:

EXAMPLE 6.1. Let $(h_N(p))_{N\geq 1}$ be a sequence of functions constructed recursively as $h_N(p) \equiv e^{h_{N-1}(p)-1}$, with $h_1(p) \equiv e^{p-1}$, and let $(g_N(p))_{N\geq 1} \equiv (ph_N(p))_{N\geq 1}$. For any given N, consider the bargaining problem where bargainer 1 has Yaari's dual preferences (Yaari, 1987) over $\mathscr{L}_e$ represented by $V^1(p, x) = g_N(p)x$ and bargainer 2 is an expected value maximizer; i.e., we can choose $V^2(p, x) = px$.

Clearly, for any bargaining problem within this family, DUEL is satisfied, and since for both bargainers preferences are separable and $V_{xx}^i = 0$, CEC* and IMCE hold as well. Therefore, for each N, the bargaining problem has a unique EMB outcome, $\bar{\mathbf{x}} = (\bar{x}, 1-\bar{x})$, where $\bar{x}$ is given by $1/(g_N'(1)\bar{x}) = 1/(1-\bar{x})$. Routine calculations show that $g_N'(1) = 2$ and $g_N''(1) = N+2$, This yields $\bar{x} = 1/3$ independently of the value of N.

Since we are in a two-person[15] bargaining situation with a unique EMB outcome, for any given N the outcome $\bar{\mathbf{x}} = (1/3, 2/3)$ is also the limit as $\rho \to 1$ of the (unique, by CEC*) SSPE proposals in the alternating-offers game. In this situation, however, for sufficiently large values of N, a Nash outcome does not exist, which we illustrate later. By Proposition 6.1, it suffices to consider whether $\bar{\mathbf{x}} = (1/3, 2/3)$ can be successfully appealed. Since bargainer 2 is an expected value maximizer, bargainer 1 can appeal against $\bar{\mathbf{x}}$ if there is a small concession $s > 0$ and probability $1-q$ with (i) $(2/3)(1-q) = 2/3 - s$ such that (ii) $1/3 < g_N(1-q)(1/3+s)$. Figures 2(a) and 2(b) plot the line (i) and the upper contour set (ii), with $p = 1-q$ and $x = 1/3+s$ for

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(i) a Nash outcome exists for each couple of preferences in the set, and (ii) for preferences which do not belong to the set there exists another preference relation (which can be chosen from the set) such that the resulting bargaining problem has no Nash outcome.

[14] This implication follows from Corollary 2 in Chew, Karni, and Safra (1987).

[15] One can readily verify that g_N exhibits increasing elasticity for all N, so DICE is also satisfied. Thus, the example can be readily extended to a situation with $I > 2$ bargainers.

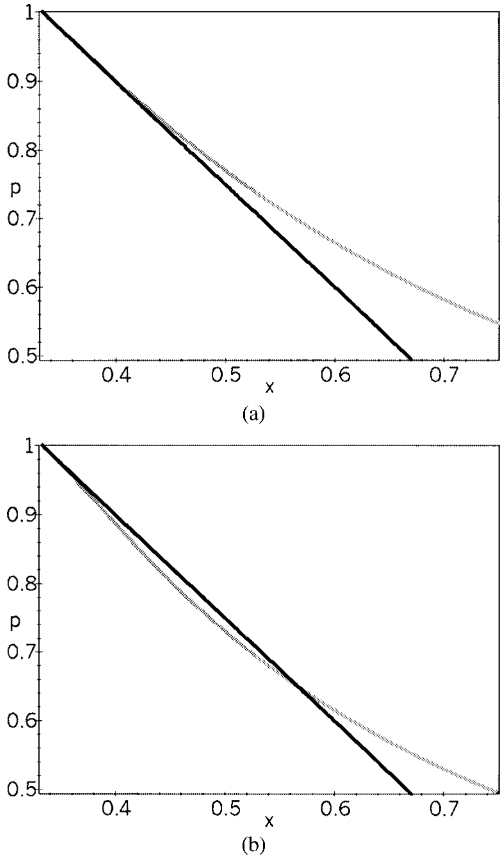

FIG. 2. Examples with multiplicatively separable preferences where (a) the unique EMB outcome is also an ordinal Nash outcome, and (b) the unique EMB outcome is *not* an ordinal Nash outcome.

the cases $N = 6$ and $N = 15$, respectively. It turns out that if $N \leq 6$, then the upper contour set (ii) lies above (i), and thus $\bar{\mathbf{x}}$ is a Nash outcome. If $N > 6$, however, then there does exist a proposal which is preferred to receiving $\bar{x} = 1/3$ for certain.[16]

[16]Interestingly, the preferences of bargainer 1 violate Hanany and Safra's necessary and sufficient conditions for the existence for the existence of Nash outcomes for *all* N. Thus, for any given N, there must be some preferences for bargainer 2 such that the associated bargaining problem has no Nash outcome. We showed earlier that when $N > 6$, it is enough to take bargainer 2 to be an expected value maximizer, but that when $N \leq 6$, the challenge to find such preferences remains.

6.2. *Comparison with Safra and Zilcha's Extensions*

Safra and Zilcha's *functional Nash solution* associates with a bargainer problem $(\succeq^i)_{i=1}^{i=I}$ the outcome in $\mathbf{X}$ that maximizes the product $\prod_{i=1}^{I}(U^i - d^i)$ for $U^i \geq d^i$, where each U^i is a smooth functional over marginal cumulative distribution functions over $\mathbf{X}$ that represents preferences $\succeq^i$ and $d^i = U^i(\delta_D)$. The functional Nash solution is well defined provided that all U^i representing a given $\succeq^i$ are related by positive affine transformations. This is the case with, among others, the families of DL and RDEU preferences. We have seen that in both cases, we can represent preferences $\succeq^i$ over $\mathscr{L}_e$ by taking $V^i(p, x) = g^i(p)u^i(x)$ with $g^i(0) = 0, g^i(1) = 1$, $u^i(0) = 0$, and $u^i(1) = 1$, and that we can choose $g^i(p) = p$ in the DL case. For the case in which the bargaining set $\mathbf{X}$ is the one-dimensional simplex, if preferences $\succeq^i$ display risk aversion and $D \sim^i 0$, then the functional generalization of Nash solution for both DL and RDEU preferences is characterized by the maximization program

$$\mathbf{x} = \arg\max_{y \in \mathbf{X}} \prod_{i=1}^{I} u^i(y^i). \tag{9}$$

Comparing (9) with (4), we see that for the class of DL preferences, this extension coincides with the EMB solution. However, for the class RDEU preferences, the EMB solution generally differs from the functional Nash solution.

A local utility approach also suggested in Safra and Zilcha's paper leads to a different extension of the Nash solution to general (smooth) preferences: the *sequential Nash solution.* Since this extension coincides with the functional generalization for the case of RDEU preferences, it also differs from the EMB solution.

APPENDIX

Proof of Proposition 3.2

We show that the determinant of the matrix (2) is positive. By relabeling row and column index k of (2) to $I - k + 1$, it suffices to show the following (throughout, by convention, if $i > n$, then index i indicates the i–nth element).

THEOREM A.1. *Let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix such that for all i, $a_{ii} > a_{i+1i} > \cdots > a_{i-1i} > 0$. Then* $\det(A) > 0$.

Proof. The result holds by the following three lemmas.

LEMMA A.2. *Let* $A = (a_{ij})_{i,j=1}^n$ *be an* $n \times n$ *matrix such that for all* i, $1 = a_{ii} \geq a_{i+1i} \geq \cdots \geq a_{i-1i} > 0$. *Then there exists a matrix* $B = (b_{ij})_{i,j=1}^n$ *with the properties* (i) $b_{ij} = 1$ *or* 0 *for any* i, j; (ii) $1 = b_{ii} \geq b_{i+1i} \geq \cdots \geq b_{i-1i}$; *and* (iii) $\det(A) \geq \det(B)$.

Proof of Lemma A.2. Denote by A_{ij} the (i, j) cofactor of matrix A; i.e., A_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix which is obtained from A by eliminating the ith row and the jth column, multiplied by $(-1)^{i+j}$.

Consider the following procedure on matrix A. If there is a column such that $1 > a_{ij} = a_{i+1j} = \cdots = a_{kj} > 0$ for $i < k$, then if $A_{ij} + A_{i+1j} + \cdots + A_{kj} \geq 0$, replace $a_{ij}, a_{i+1j}, \ldots, a_{kj}$ with $a_{k+1,j}$ (which is less than these elements by assumption); otherwise, replace them with $a_{i-1,j}$. If there is no such column, then every element of A is either 1 or 0, and hence A already has the required property. Observe that this procedure strictly increases the number of common elements in column j. Thus repeating this procedure finitely many times, every element of the matrix becomes either 1 or 0, and call it B. From the Laplace expansion, it readily follows that this procedure always decreases the determinant and preserves the sign condition of the original matrix. Therefore, B satisfies the required properties. ■

LEMMA A.3. *If matrix* B *satisfies* (i) *and* (ii) *in Lemma* A.2, *then there exists a column* k *such that the sign of the determinant does not change if all of the elements of column* k *are replaced with* 1.

Proof of Lemma A.3. There is nothing to prove if there is a column where every element is 1. If there is not, then choose a sequence of indices $t_k, k = 1, \ldots, n+1$, by the following rule: $t_1 = 1$, t_k is the first $t_k > t_{k-1}$ with $b_{t_k t_{k-1}} = 0$. Since $t_k \in \{1, \ldots, n\}$, there are k and $k', k \neq k'$, with $t_k = t_{k'}$ and $t_k, \ldots, t_{k'-1}$ are distinct. Denote by $b(j)$ the jth column vector of B. We claim that $b(t_k) + \cdots + b(t_{k'-1}) = c\mathbf{e}$, where c is a positive integer and $\mathbf{e} = (1, \ldots, 1)'$. To see this, note that by construction, if the ith element of $b(t_j)$ is 1, then the $i+1$th element of $b(t_j)$ is either 1 or 0 (thus $t_{j+1} = i+1$) and the ith element of $b(t_{j+1})$ is 0 and the $i+1$th element of $b(t_{j+1})$ is 1. So for any row i, 1 appears exactly the same times as it does in row $i+1$; hence the claim follows. Since adding columns $t_{k+1}, \ldots, t_{k'-1}$ to column t_k does not change the determinant, we have $\det(B) = \det([\ldots, c\mathbf{e}, b(t_k+1), \ldots]) = c\det([\ldots, \mathbf{e}, b(t_k+1), \ldots])$, and so column t_k has the desired property. ■

LEMMA A.4. *If matrix* B *satisfies* (i) *and* (ii) *in Lemma* A.2, *then* $\det(B) \geq 0$.

Proof of Lemma A.4. We prove this by induction. It holds for $n = 1$. Suppose that it is also true for $n-1$ and $n > 1$. By Lemma A.3, we can set one of the columns, say $b(l)$, equal to $\mathbf{e}$. Since we can relabel columns and rows so that properties (i) and (ii) in Lemma A.2 are met, assume $l = 1$ without loss of generality. If there is another column that is equal to $\mathbf{e}$, then $\det(B) = 0$, and we are done. If there is not, then pick t_k, $k = 1, \ldots, n+1$, as in the previous proof, starting with $t_1 = 2$. If $\{t_k\}$ constitutes a loop before $t_k = 1$ occurs, then the same argument shows that we can replace a column other than the first with $\mathbf{e}$ without changing the sign of the determinant, and so we are done. Otherwise, let k be the first time $t_k = 1$. Then in the expression $b(t_1) + \cdots + b(t_{k-1})$, 1 appears exactly the same number of times (say α times) in every row, except one less time in the first row.

Now multiply columns $t_2, \ldots, t_{k-1}$ by $(1/\alpha)$ and subtract them from column $b(1)$. This operation does not change the determinant, and the resulting column vector is 0 for the off-diagonal elements, and the diagonal element is $1 - \frac{\alpha-1}{\alpha} > 0$. By the Laplace expansion with respect to column 1 and the induction hypothesis, the determinant of the resulting matrix is nonnegative. Thus $\det(B) \geq 0$ holds for n. ■

Proof of Proposition 4.1

Under the assumption on the V^i, $b^i(\mathbf{x}) = \frac{V^i_x(1,x^i)}{V^i_p(1,x^i)}$ is bounded away from 0 for $\mathbf{x} \in \Delta$ and $b^i(\mathbf{x}) = \frac{V^i_x(1,x^i)}{V^i_p(1,x^i)} \to \infty$ as $x^i \to 0$. For each $\mathbf{x} \in \Delta$, let $\bar{b}(\mathbf{x}) = (\frac{1}{\sum_{i=1}^I b^i(\mathbf{x})})(b^i(\mathbf{x}))_{i=1}^I$; i.e., $\bar{b}$ is the normalized vector of marginal boldness. By construction, $\mathbf{x}$ is an EMB outcome if $\bar{b}^i(\mathbf{x}) = \frac{1}{I}$ for every bargainer i.

For every $k = 1, \ldots$, let $\mathbf{X}_k = \{\mathbf{x} \in \mathbf{X} : x^i \geq \frac{1}{(k+1)I}$ for all $i\}$ and consider a correspondence Ξ_k from $\mathbf{X}_k$ to itself given by

$$\Xi_k(\mathbf{x}) = \left\{\mathbf{y} \in \mathbf{X}_k : y^i = \frac{1}{(K+1)I} \text{ if } \bar{b}^i(\mathbf{x}) < \frac{1}{I}\right\}.$$

That is, $\Xi_k(\mathbf{x})$ assigns the worst available outcome if bargainer is not bold enough. Since $\sum_{i=1}^I \bar{b}^i(\mathbf{x}) = 1$, Ξ_k is nonempty valued, and it is readily verified that it is convex valued with closed graph. So by Kakutani's fixed point theorem, there is a fixed point $\bar{\mathbf{x}}_k \in \mathbf{X}_k$, i.e., $\bar{\mathbf{x}}_k \in \Xi_k(\bar{\mathbf{x}}_k)$, for each k. Abusing notation, we write $\{\bar{\mathbf{x}}_k : k = 1, \ldots\}$ for a convergent subsequence, and let $\bar{\mathbf{x}} \in \mathbf{X}$ be its limit point.

It $\bar{x}^i = 0$ for some i, then $\sum_{i=1}^I b^i(\bar{\mathbf{x}}_k) \to \infty$ as $k \to \infty$ by the boundary property of b^i. So for any j with $\bar{x}^j > 0$, we have $\bar{b}^j(\bar{\mathbf{x}}_k) \to 0$ as $k \to \infty$. So $\bar{x}^j_k = \frac{1}{(k+1)I}$ for all k large enough, since $\bar{\mathbf{x}}_k \in \Xi_k(\bar{\mathbf{x}}_k)$, but then $\bar{x}^j > 0$ could not be the limit. So we conclude that $\bar{\mathbf{x}} >> 0$. Then for all k large enough, $\bar{x}^i_k > \frac{1}{(k+1)I}$ must hold for every i, which is possible only if $\bar{b}^i(\bar{\mathbf{x}}_k) \geq \frac{1}{I}$ for every i. Since $\sum_{i=1}^I \bar{b}^i(\bar{\mathbf{x}}_k) = 1$, $\bar{b}^i(\bar{\mathbf{x}}_k) = \frac{1}{I}$ must hold for every i, if k is large enough. ■

Nonexistence of Ordinal Nash Outcome if We Allow for Simultaneous Concessions When $I > 2$

We show that no outcome is robust against bargainers seeking concessions *simultaneously* from at least two of the other bargainers. If such a robust outcome exists, then it must be an EMB outcome, so let $\mathbf{x} \in \Delta$ be an EMB outcome. We show that $\mathbf{x}$ is not robust against bargainer 1 demanding an extra share. By the implicit function theorem, for each $j, I \geq j \geq 2$, there is a C^1 function $s^j(q)$ defined for small $q \geq 0$ such that $V^j(1-q, x^j) = V^j(1, x^j - s^j(q))$ for all q, and $s^j_q(0) = V^j_p(1, x^j)/V^j_x(1, x^j) = 1/b^j(\mathbf{x})$. Now $\frac{d}{dq}V^1(1-q, x^1 + \sum_{j=2}^I s^j(q))|_{q=0}$ yields $-V^1_p(1, x^j) + V^1_x(1, x^j)\sum_{j=2}^I s^j_q(0) = -V^1_x(1, x^j)[\frac{1}{b^1(x)} - \sum_{j=2}^I \frac{1}{b^j(x)}]$. Observe that since $b^j(\mathbf{x}) = b^1(\mathbf{x})$ for all j, then this expression is strictly positive, unless $I = 2$. This implies that bargainer 1 can successfully offer $s^j(q)$ to each j simultaneously, for small enough probability of breakdown q, if $I > 2$.

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