# Decomposable Choice under Uncertainty[1]

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Savage motivated his sure–thing principle by arguing that, whenever an act
would be preferred if an event obtains and preferred if that event did not obtain,
it should be preferred overall. The ability to decompose and recompose decision
problems in this way has normative appeal. It does not, however, require the full
separability embodied in Savage's axiom. We formulate a weaker axiom that suf-
fices for decomposability, and show it is almost equivalent to Gul and Lantto's
dynamic programming solvability property. Given probabilistic sophistication,
weak decomposability is equivalent to betweenness. Without probabilistic
sophistication, weak decomposability implies an implicit additive representation.
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gramming solvability; nonexpected utility; betweenness.

## 1. SAVAGE AND DECOMPOSING CHOICE PROBLEMS

In Savage's axiomatization of subjective expected utility theory under uncertainty, his second postulate P2, often referred to as the sure-thing principle, is the analogue of the independence axiom from standard expected utility theory under objective risk. Thus, theoretical and empirical criticisms of the separability assumptions implicit in the independence axiom apply equally to P2.[2] Savage originally motivated the sure-thing principle, however, with an example that contains no explicit mention of separability.

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter for himself, he asks whether he would buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. [24, p. 21]

Decisions like buying property are difficult in part because each choice can give very different outcomes in different states of the world. The businessman's thought-experiment, breaking each alternative into component parts and comparing event by event, is a common way to simplify such problems. For this method to work, such decomposition and recomposition must yield the correct final decision: that is, if buying the property is preferred in all contingencies then it should be preferred overall. As Savage argued, this seems both a plausible and useful restriction to place on preferences: "except possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance."

Savage used the appeal of this idea to justify his second postulate. Indeed, it was this idea, rather than P2 itself, that Savage first referred to as the sure-thing principle. In this paper, however, we argue that one can accept the idea of Savage's story while rejecting the separability assumptions

embodied in Savage's postulate P2. Specifically, we propose a weaker rule —a weak decomposability principle—that allows the kind of decomposition and recomposition of decision problems used in the property-buying example, but that does not imply full separability across events.

How then does Savage get from the nice property-buying story to the controversial separability implications of P2? The answer lies in his formalization. In Savage's framework, acts are functions from states of the world to outcomes. We can think of buying a property as an act, $g$, yielding different final outcomes depending on the state of the world. For the purpose of the example, not buying the property is also considered an act. Since it is hard to envisage an act defined negatively, let us interpret the property in question as a house and define the act $f$ as "not to buy and to stay put in the old residence." Paraphrasing Savage, we can write the idea of decomposability as: if a person prefers an act $g$ to another act $f$, either knowing that the event $B$ obtains, or knowing that the complement of the event $B$ obtains, then that person should prefer $g$ to $f$.

So far, so good. For Savage, however, preferences are only defined on the set of acts, and acts are functions from *all* states to outcomes. Objects like "$g$ if the event $B$ obtains" are, at best, subacts. To formalize the statement that $g$ is preferred to $f$ if $B$ obtains, Savage needs to extend these subacts over the whole domain. This was his next step.

> What technical interpretation can be attached to the idea that $g$ would be preferred to $f$, if $B$ were known to obtain. Under any reasonable interpretation, the matter would seem not to depend on the values $f$ and $g$ assume at states outside of $B$. There is, then, no loss of generality in supposing that $f$ and $g$ agree with each other except in $B$; ... . The first part of the sure-thing principle can now be interpreted thus: If after being modified so as to agree with one another outside of $B$, $g$ is preferred to $f$; then $g$ would be preferred to $f$ if $B$ were known. [24, p. 22]

This "technical interpretation" is less innocent than it at first appears. Consider, for example, the three acts, "to buy the house," "not to buy the house (and to stay put)," and "to emigrate to Japan." Savage interprets the statement that "the agent would prefer to buy over not to buy if he knew the Republican would win" to mean that the agent prefers the act, say, "buy if the Republican wins, and emigrate to Japan otherwise" to the act "do not buy (and stay put) if the Republican wins, but still emigrate to Japan otherwise." Moreover, the same preference must obtain if we change "emigrate to Japan" to any other activity, regardless of its outcomes. But, it is precisely this separability—that preferences on the event $B$ do not depend on what happens off $B$—that has been challenged on experimental and introspective grounds. For example, our hero might anticipate that his

attitude toward owning and living in the house might be quite different if he knew that it came to him in place of a life in Japan. Compared to emigration, the house might now seem small and confining. Alternatively, compared to typical Japanese real estate, the house might now seem large and more attractive. Such thinking contradicts the separability implicit in Savage's particular "technical interpretation" of the idea in the story. But, Savage's technical interpretation is not necessary to capture this idea.

How else can one reasonably interpret the statement that "the agent would prefer to buy the house if he knew the Republican would win?" We take it simply to mean that the agent prefers the act "to buy the house if the Republican wins, and not to buy otherwise" to the act "not to buy the house regardless of the election results." Under this interpretation, the statement has no implications about preferences among elaborate acts involving trips to Japan in the event of a Democratic victory: it only concerns preferences over acts constructed from the original acts "to buy" or "not to buy" the house. An axiom based on this interpretation, however, is sufficient to carry out the decomposition and recomposition used in the businessman's thought-experiment. In considering whether or not to buy the house, the businessman first compares the act "to buy if the Republican wins but not otherwise" with the act "not to buy regardless of the election." He then compares the act "to buy if the Democrat wins but not otherwise" with the act "not to buy regardless." Our axiom simply says that, if in both comparisons, the former is better than the latter, then he should prefer the act "to buy regardless" to the act "not to buy regardless." That is, our axiom directly formalizes the idea in Savage's example under our less restrictive interpretation of statements involving preferences over sub-acts. We call this axiom *weak decomposability*. The purpose of this paper is to explore the implications of this axiom: what representations are consistent with decomposability, and how it relates to other properties of preference relations.

Section 2 formally introduces weak decomposability, discusses what it allows and does not allow, and shows that it is almost equivalent to a version of Gul & Lantto's [17] dynamic programming solvability property translated to a Savage framework. Section 3 assumes that the agent is probabilistically sophisticated; she treats uncertainty as if it were risk. In this case, weak decomposability is equivalent to Chew's [2, 3] and Dekel's [7] betweenness property. Thus, just as Savage's original sure thing property forms part of the axiomatization of subjective expected utility theory, so weak decomposability forms part of an axiomatization of subjective betweenness theory.[3] Section 4 provides more general representation

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[3] Epstein [10] points out that the betweenness property is sufficient for many standard results in optimal choice.

results that do not rely on probabilistic sophistication. Weak decomposability still ensures that preferences admit at least an implicit additive representation.[4] Section 5 briefly discusses the relation to Skiadas's [27, 28] "conditional preference" approach. Unless stated otherwise, proofs are to be found in the appendix.

## 2. THE WEAK DECOMPOSABILITY PRINCIPLE

*Setup and notation.* Denote by $\mathscr{S} = \{..., s, ...\}$ a set of states, $\mathscr{E} = \{..., A, B..., E, ...\}$ the set of events which is a given $\sigma$-field on $\mathscr{S}$, and $\mathscr{X} = \{..., x, y, z, ...\}$ a set of outcomes or consequences. An act is a (measurable) function $f: \mathscr{S} \to \mathscr{X}$. Let $f(\mathscr{S}) = \{f(s) \mid s \in \mathscr{S}\}$ be the outcome set associated with the act $f$, and let $\mathscr{F} = \{..., f, g, h, ...\}$ denote the set of simple acts on $\mathscr{S}$; that is, those with finite outcome sets. We will abuse notation and use $x$ to denote both the outcome $x$ in $\mathscr{X}$ and the constant act with $f(\mathscr{S}) = \{x\}$. Let $\geqslant$ be a binary relation over ordered pairs of acts in $\mathscr{F}$, representing the individual's preferences. Let $\succ$ and $\sim$ correspond to strict preference and indifference, respectively. Denote by $\geqslant_{\mathscr{X}}$ the relation over ordered pairs of outcomes obtained from $\geqslant$ for constant acts (that is, $x \geqslant_{\mathscr{X}} y$ if and only if $x \geqslant y$).

The following notation to describe an act will be convenient. For an event $E$ in $\mathscr{E}$, and any two acts $f$ and $g$ in $\mathscr{F}$, let $f_E g$ be the act which gives, for each state $s$, the outcome $f(s)$ if $s$ is in $E$ and the outcome $g(s)$ if $s$ is in the complement of $E$ (denoted $\mathscr{S} \backslash E$). In general, for any finite partition $\mathscr{P} := \{A_1, ..., A_n\}$ of $\mathscr{S}$ and any list of of $n$ acts $(h^1, ..., h^n)$, let $h^1_{A_1} h^2_{A_2} \cdots h^{n-1}_{A_{n-1}} h^n$ be the act that yields $h^i(s)$ if $s$ is in $A_i$. Using this notation, we say that an event $E$ in $\mathscr{E}$ is null with respect to the preference relation $\geqslant$ if and only if, for all acts $f$, $g$ and $h \in \mathscr{F}$, $f_E h \sim g_E h$. Let $\mathscr{N} \subset \mathscr{E}$ denote the set of such null events. We say that a finite partition $\mathscr{P}$ is non-null with respect to the preference relation $\geqslant$ if and only if every element of $\mathscr{P}$ is non-null with respect to $\geqslant$.

Savage's first six postulates, together with Machina & Schmeidler's [22] names for them, are as follows:

P1 (Ordering):   The preference relation $\geqslant$ is complete, reflexive and transitive.

P2 (Sure-thing principle):   For all events $E$ and acts $f$, $g$, $h$ and $h'$, if $f_E h \geqslant g_E h$ then $f_E h' \geqslant g_E h'$.

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[4] Formally, though this results maintains only Savage's order and monotonicity axioms, it uses stronger continuity axioms.

P3 (Eventwise monotonicity): For all non-null events $E$, pairs of outcomes $x$ and $y$, and acts $h$, $x_E h \succeq y_E h$ if and only if $x \succeq_{\mathscr{X}} y$.

P4 (Weak comparative probability): For all events $A$ and $B$, and outcomes $x^* \succ_{\mathscr{X}} x$, and $y^* \succ_{\mathscr{X}} y$, if $x_A^* x \succeq x_B^* x$ then $y_A^* y \succeq y_B^* y$.

P5 (Nondegeneracy): There exist outcomes $x$ and $y$ such that $x \succ y$.

P6 (Small event continuity): For any pair of acts $f \succ g$ and outcome $x$, there exists a finite set of events $\{A_1, ..., A_K\}$ forming a partition $\mathscr{P}$ of $\mathscr{S}$, such that for all $A$ in $\mathscr{P}$, $x_A f \succ g$ and $f \succ x_A g$.

In what follows, we will always assume (stated or otherwise) that preferences satisfy Savage's ordering assumption, P1. We will also always assume P5, not because it is necessary for the results but because the problem is trivial without it. We will not assume the other postulates unless explicitly stated.

For some results in this section we will use the following strengthening of P6 that is implied by Savage's six postulates.

P6* (Event continuity): For all acts $f$, $g$, and $h$ in $\mathscr{F}$, all outcomes $x$, $z$ in $\mathscr{X}$, and all events $A$ in $\mathscr{E}$, if $f_A h \succ g_A h$ and $x \succeq_{\mathscr{X}} y \succeq_{\mathscr{X}} z$ for all $y$ in $f(A) \cup g(A)$, then there is an event $E \subseteq A$ such that $x_E g_{A \setminus E} h \sim f_A h$ and an event $E^* \subseteq A$, such that $z_{E^*} f_{A \setminus E^*} h \sim g_A h$.[5]

For a finite partition $\mathscr{P} := \{A_k : k = 1, ..., K\}$ of $\mathscr{S}$, let $\mathscr{F}^{\mathscr{P}}$ denote the set of acts that are measurable with respect to $\mathscr{P}$. This set is naturally identified with $\mathscr{X}^K$. In section 4, we take the underlying outcome set $\mathscr{X}$ to be a compact interval of the real line. In this setting, $\mathscr{X}^K$ is a subset of $\mathbb{R}^K$, and the following third notion of continuity is natural.

P6** (Outcome continuity): For every finite partition $\mathscr{P}$ that is non-null (with respect to $\succeq$), the preference relation $\succeq$ induces a continuous relation on $\mathscr{F}^{\mathscr{P}}$.

In Section 3, we use Epstein & Le Breton's [11] strengthening of Savage's weak comparative probability axiom, P4.

P4$^c$ (Conditional weak comparative probability): For all events $T$, $A$ and $B$, $A \cup B \subseteq T$, outcomes $x^*$, $x$, $y^*$, and $y$, and acts $g$, if $x_T^* g \succ x_T g$ and $y_T^* g \succ y_T g$, then $x_A^* x_{T \setminus A} g \succeq x_B^* x_{T \setminus B} g$ implies $y_A^* y_{T \setminus A} g \succeq y_B^* y_{T \setminus B} g$.

*Decomposability.* As discussed in the introduction, our approach in this paper is to replace P2 with the following axiom that is strong enough to capture the intuition behind Savage's thought-experiment but not so strong as to imply that preferences are separable across events.

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[5] Notice that $A$ can be neither empty nor null since $f_A h \succ g_A h$. We think it is not known whether P6* is implied by P1, P3 and P6, even when $\mathscr{E}$ is the set of all subsets of $\mathscr{S}$.

*Weak decomposability.*  For any pair of acts $f$ and $g$ in $\mathscr{F}$, and any event $A$ in $\mathscr{E}$: $g_A f \succ f$ and $f_A g \succ f$ implies $g \succ f$.

In words, starting from the act $f$, if the agent is made better off by substituting $g$ for $f$ on $A$ and she is also made better off by substituting $g$ for $f$ on $\mathscr{S} \backslash A$, then she unconditionally prefers $g$ to $f$. This axiom simply formalizes our interpretation of the example in the introduction, where $f$ is the act "not to buy the house," $g$ is the act "to buy," and $A$ is the event of a Republican victory in the coming election.

Weak decomposability is weaker than Savage's sure-thing principle but it still places some restrictions on preferences. To get some idea what weak decomposability allows and what it rules out, consider acts constructed from two "base" acts $f$ and $g$, and two (disjoint and exhaustive) events $A$ and $B$. There are six strict preferences that rank $f = f_A f$ bottom:

$$(1)\quad g_A g \succ g_A f \succ f_A g \succ f_A f \qquad (2)\quad g_A g \succ f_A g \succ g_A f \succ f_A f$$

$$(3)\quad g_A f \succ g_A g \succ f_A g \succ f_A f \qquad (4)\quad f_A g \succ g_A g \succ g_A f \succ f_A f$$

$$(5)\quad g_A f \succ f_A g \succ g_A g \succ f_A f \qquad (6)\quad f_A g \succ g_A f \succ g_A g \succ f_A f$$

Rankings (1) and (2) are consistent with both weak decomposability and the sure-thing principle. Rankings (3) and (4) are still consistent with weak decomposability, but they violate the sure-thing principle. For example, in ranking (3), $h_A g \succcurlyeq h_A f$ when $h = f$, but $h_A f \succ h_A g$ when $h = g$. Rankings (5) and (6) violate both weak decomposability and the sure-thing principle. For example, in ranking (5), $f_A g \succ g_A g$ and $g_A f \succ g_A g$ but $g_A g \succ f_A f$.

The extra rankings permitted by weak decomposability include some we might want to allow. For example, suppose that the event $B$ is partitioned into $B_1$ and $B_2$, that the event $A$ is "large," and the event $B_2$ is "small." Let $g$ be the act that gives \$100 on $A$, \$500 on $B_1$, and \$0 on $B_2$. Let $f$ be the act that gives \$0 on $A$, and \$100 on $B$. Then, $g_A f$ is the act that yields \$100 dollars everywhere, and $f_A g$ is the act that yields \$500 on $B_1$ but \$0 elsewhere. In this case, ranking (3)—allowed by weak decomposability but ruled out by the sure-thing principle—corresponds to the preferences usually found in Allais-type experiments.

The rankings ruled out by weak decomposability are perhaps more associated with fair-allocation problems than with individual choice under uncertainty. For example, consider Diamond's [8] problem of a mother allocating an indivisible gift to one of her two children, Fred and Gail. If forced to choose, the mother prefers Gail to Fred, but (perhaps for fairness reasons) she would prefer to "randomize" by making the allocation depend on the result of a cricket test match between Australia and England. Let $A$ be the (more likely) event that Australia wins the match. Then, in the obvious notation, she might prefer $g_A f$ to $f_A g$ but prefer either of these to

giving the gift outright to either child. These preferences correspond to ranking (5) above, ruled out by weak decomposability. Loosely speaking, the problem here is that the subact $f$ on $A$ is a "complement" of the subact $g$ on $S\setminus A$ (and vice versa). Weak decomposability rules out this degree of "complementarity" across events.[6]

*Equivalent notions.* Although weak decomposability is written for a simple decomposition of the state space into two events, $A$ and $S\setminus A$, the idea automatically extends to any finite partition.[7] Moreover, if $\geqslant$ satisfies weak decomposability then for any pair of disjoint events $B$ and $C$ in $\mathscr{E}$ and all acts $f$, $g$ and $h$ in $\mathscr{F}$: $g_B f_C h \succ f_{B \cup C} h$ and $f_B g_C h \succ f_{B \cup C} h$ implies $g_{B \cup C} h \succ f_{B \cup C} h$. This is a particularly natural property to require if we regard the parent preferences, $\geqslant$, themselves to be derived from "grandparent" preferences, where "updating" has resulted from the resolution of some earlier (but unmodelled) uncertainty.[8]

Returning to our interpretation of Savage's thought experiment, the reader might object that we gave a special "status-quo" role to the act "not to buy the house." For example we interpreted the statement the agent would prefer to buy the house if he knew the Republican would win to mean he prefers the act "to buy if the Republican wins but not to buy otherwise" to the act "not to buy regardless." An equally reasonable interpretation is that he prefers the act "to buy regardless" over the act "not to buy if the Republican wins, but to buy otherwise." The corresponding weak decomposability axiom would impose that, if in addition he prefers "to buy regardless" over "to buy if the Republican loses but not to buy otherwise" then the businessman should prefer the act "to buy regardless" to the act "not to buy regardless." That is, starting from the act $g$, if the agent is made worse off by substituting $f$ for $g$ on $A$ and she is also made worse off by substituting $f$ for $g$ on $\mathscr{S}\setminus A$, then she unconditionally prefers $g$ to $f$. The following proposition, however, shows that, given the Savage framework notions of "monotonicity" and "continuity," these two versions of weak

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[6] The use of a random mechanism to allocate an indivisible good in an equitable or fair way is an old idea. For example it appears in Hobbes [18, Chapter XV, p. 165]. We thank Mamoru Kaneko for bringing this reference to our attention. Johnsen & Donaldson [19] consider perverse preferences rather like these in a dynamic context. They use what they call "conditional weak independence" to rule them out. Peter Klibanoff suggested the term "complementarity."

[7] Formally: suppose that preferences, $\geqslant$, satisfy P1 and weak decomposability. Then, for all pairs of acts $f$ and $g$, and any finite partition of the state space $\{E_1, ..., E_n\} \subset \mathscr{E}$, if $g_{E_i} f \succ f$ for all $i = 1, ..., n$ then $g \succ f$.

[8] Here, "updated preferences" refer to the preferences on the event $B \cup C$, holding fixed the act $h$ on the complement to this event. This notion of updating does not require separability. Epstein & Le Breton [11] and Sarin & Wakker [23] argue that such updated preferences should still respect axioms imposed on parent preferences.

decomposability are equivalent. Moreover, we can express the idea using weak preference, strict preference, and even (given P6* continuity) indifference.

LEMMA 1. *Suppose that Savage postulates* P1, P3, *and* P6 *hold. Then weak decomposability is equivalent to any of the following statements*: *for any pair of acts f and g in $\mathscr{F}$, and any event A in $\mathscr{E}$*:

1. $g_A f \succcurlyeq f$ and $f_A g \succcurlyeq f$ imply $g \succcurlyeq f$;
2. $g_A f \succ f$ and $f_A g \succcurlyeq f$ imply $g \succ f$;
3. $g \succ f_A g$ and $g \succ g_A f$ imply $g \succ f$;
4. $g \succcurlyeq f_A g$ and $g \succcurlyeq g_A f$ imply $g \succcurlyeq f$;
5. $g \succ f_A g$ and $g \succcurlyeq g_A f$ imply $g \succ f$.

*Furthermore*, *weak decomposability implies*:

6. $g_A f \sim f$ and $f_A g \sim f$ imply $g \sim f$,[9]

*and this is equivalent to weak decomposability if in addition* P6* *holds*.

Gul & Lantto [17] consider normative rules for an agent's choices within and between dynamic decision trees. Their aim, following Machina [20], is to weaken the standard consequentialist assumption, while still retaining some degree of consistency across choices within trees. Among the normative restrictions they suggest is a property they call dynamic programming solvability (DPS).[10] To illustrate the idea, they give the example of an agent who has to decide how to go to work. The options are to walk, to drive, to bike, or to take the bus. Suppose the following two plans are optimal: (1) drive if it rains, bike if it is sunny; and (2) take the bus if it rains, walk if it is sunny. Then, they argue, the following plans of actions should also be optimal: (3) drive if it rains, walk if it is sunny; and (4) take the bus if it rains, bike if it is sunny. That is, dynamic programming solvability implies that "shuffling" two optimal plans of actions produces another optimal plan of action.

This seems a desirable property for preferences to have in dynamic decision problems. Unlike consequentialism, it does not imply that the agent's choice at each final decision node is independent of what her choices would have been at other, unrealized, nodes. But it does mean that, if there is more than one optimal overall plan, her choice at each final decision node

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[9] Property 6 can be regarded as an analogue to Chew & Epstein's [4] "indifference separability" property, part of their axiomatization of betweenness preferences under risk.

[10] Gul & Lantto show that dynamic programming solvability is equivalent to two other weakenings of consequentialism in the context of dynamic choice, but we do not know how to translate these into a Savage framework.

does not depend on which optimal plan she would have followed at the other nodes. In this sense, DPS is a decomposition property, albeit in another context, and it would be nice if it were related to weak decomposability. To explore any relation, however, we have first to translate their axiom on dynamic choices under objective risks to an axiom on static preferences under subjective uncertainty. We trust that the following property captures at least some of Gul & Lantto's original intuition.

   DPS*: *The preference relation* $\geqslant$ *satisfies* DPS* *if, for any finite partition* $\mathscr{P} := \{A_1, ..., A_n\}$ *of the state space* $\mathscr{S}$, *and for any pair of acts* $f$ *and* $g$ *in* $\mathscr{F}$ *with* $f \sim g$: *if* $f \geqslant h$ *for all acts* $h$ *such that, for each event* $A_j$ *in* $\mathscr{P}$, *either* $h(s) = f(s)$ *for all* $s$ *in* $A_j$ *or* $h(s) = g(s)$ *for all* $s$ *in* $A_j$, *then* $f \sim h$ *for all such acts* $h$.

   PROPOSITION 2. *Suppose that a preference relation* $\geqslant$ *satisfies* P1, P3, *and* P6 *and* P6*.[11] *Then the preference relation satisfies* DPS* *if and only if it satisfies weak decomposability*.

   This equivalence result does not require the agent to be probabilistically sophisticated, or even to satisfy Savage's P4 axiom. We next consider the implications of weak decomposability in the presence of probabilistic sophistication.

## 3. REPRESENTATION WITH PROBABILISTIC SOPHISTICATION

   Savage's original sure thing principle was part of an axiomatization not only of separable preferences over lotteries (expected utility) but also of additive subjective beliefs (probabilistic sophistication). Recent work by Machina & Schmeidler [22] and others[12] has shown that additive subjective beliefs can be axiomatized without requiring the agent's preferences over lotteries to obey the expected utility hypothesis. Formally they define probabilistic sophistication as:

   Probabilistic Sophistication. *A preference relation is said to be* probabilistically sophisticated, *if there exists a finitely additive probability measure* $\mu$ *on* $\mathscr{E}$ *such that for any pair of acts* $f$ *and* $g$, *if* $\mu \circ f^{-1}(x) = \mu \circ g^{-1}(x)$ *for all* $x$ *in* $f(\mathscr{S}) \cup g(\mathscr{S})$ *then* $f \sim g$.

   Probabilistic sophistication means that we can represent the individual's beliefs over the states of the world with a probability measure and,

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[11] The extra continuity property, P6* is only used to show DPS* implies weak decomposability.

[12] See Epstein & LeBreton [11] and Grant [13].

moreover, we can separate those beliefs from her "risk preference." To see this, notice that we can use the measure $\mu$ that represents her beliefs, to map acts into $\mathscr{L}_0$, the set of lotteries with finite support, as follows:

$$f \mapsto P, \qquad \text{where} \quad P(x) = \mu \circ f^{-1}(x) \quad \text{for all } x \text{ in } f(\mathscr{S})$$

We can then identify the individual's "risk preferences" with the induced relation over lotteries with finite support; that is, we can define an induced relation $\succcurlyeq_{\mathscr{L}_0}$ over lotteries with a finite support by the rule that for any two lotteries $P$ and $Q$, $P \succcurlyeq_{\mathscr{L}_0} Q$ implies there exists two acts $f$ and $g$ such that $f \succcurlyeq g$, $\mu \circ f^{-1}(x) = P(x)$ and $\mu \circ g^{-1}(x) = Q(x)$ for all $x$ in $f(\mathscr{S}) \cup g(\mathscr{S})$. Probabilistic sophistication ensures that the induced relation is transitive. Moreover knowledge of $\succcurlyeq_{\mathscr{L}_0}$ and $\mu$ enables the analyst to recover all of $\succcurlyeq$ since for any pair of acts $f$ and $g$ which are mapped by $\mu$ to $P$ and $Q$ respectively, we may correctly infer that $f \succcurlyeq g$ if and only if $P \succcurlyeq_{\mathscr{L}_0} Q$.

The following proposition says that, if we assume probabilistic sophistication, then weak decomposability is equivalent to the Chew–Dekel betweenness property. That is, if for any two lotteries $P$ and $Q$, $P \succ_{\mathscr{L}_0} Q$, then for any probability mixture (i.e., convex combination) of the two lotteries, $\lambda P + (1 - \lambda) Q$, we have $P \succ_{\mathscr{L}_0} \lambda P + (1 - \lambda) Q$ and $\lambda P + (1 - \lambda) Q \succ_{\mathscr{L}_0} Q$.

PROPOSITION 3. *Suppose a preference relation $\succcurlyeq$ over $\mathscr{F}$ is probabilistically sophisticated, and satisfies* P1, P3, *and* P6. *Let $\mu$ be the associated finitely additive probability measure on $\mathscr{E}$. Then the following statements are equivalent*:

(i) $\succcurlyeq$ *satisfies weak decomposability.*

(ii) $\succcurlyeq_{\mathscr{L}_0}$ *is represented by a continuous utility function V which is both quasi-convex and quasi-concave in probability mixtures* (*that is*, $\succcurlyeq_{\mathscr{L}_0}$ *satisfies betweenness*).

Combining the earlier work of Chew [2, 3], Dekel [7], Machina & Schmeidler [22] and Epstein & Le Breton [11], with Proposition 3 yields the immediate corollary that weak decomposability forms part of an axiomatization of betweenness theory.

COROLLARY 4. *Suppose that the set of outcomes $\mathscr{X}$ is the compact interval $[\underline{x}, \bar{x}] \subset \mathbb{R}$, where $\underline{x} < \bar{x}$, and that $x > y$ implies $x \succ_{\mathscr{X}} y$. Then the following statements are equivalent.*

1. *The preference relation $\succcurlyeq$ over $\mathscr{F}$ satisfies* P1, P3, P4$^C$, P6 *and weak decomposability.*

2. *There exists a finitely additive, strongly-continuous probability measure*[13] $\mu$ *on* $\mathscr{E}$ *and a function* $V\colon \mathscr{F} \to \mathbb{R}$ *that represents* $\succcurlyeq$, *such that* $V$ *is implicitly defined by* $\int_{\mathscr{X}} v(x, V(f))\, \mu \circ f^{-1}(dx) = 0$ *where* $v\colon \mathscr{X} \times \mathbb{R} \to \mathbb{R}$ *is increasing in its first argument.*

For intuition, recall that Gul & Lantto's [17] original DPS property was defined assuming probabilistic sophistication. They showed that it is equivalent to betweenness. Proposition 2 above showed that, even without probabilistic sophistication, weak decomposability is equivalent to at least a version of DPS. Proposition 3 completes the triangle. Alternatively, recall that weak decomposability rules out "complementarity" across events. The linear indifference sets of betweenness preferences, similarly rule out such complementarity.

For a third intuition for this result, notice that weak decomposability implies a betweenness-like property even without probabilistic sophistication. Consider two acts $f$ and $g$, an event $E$ and an outcome $z$. Suppose we can find outcomes $x$ and $y$ such that $y \succcurlyeq x$, $f_E x \sim z$ and $g_E y \sim z$. That is, to make the subact $g$ on $E$ indifferent to $z$ we have to augment it with a (weakly) better outcome off $E$ than we do for $f$ on $E$. Then we can define a natural preference relation over the sub-acts $f$ on $E$ and $g$ on $E$ with respect to the outcome $z$, by the rule $f$ on $E$ is (weakly) preferred to $g$ on $E$ if $y \succcurlyeq x$. This induced preference relation is defined on those sub-acts on $E$ that can be "pulled onto" the indifference set of $z$. The following lemma says that, given weak decomposability, the induced preference relation satisfies P2. This is analogous to betweenness implying independence within an indifference set.

LEMMA 5. *Assume* P1, P3, P6 *and weak decomposability. Let $A$ and $B$ be disjoint, non-null events such that $(A \cup B)^c$ is also non-null. Suppose that $f_A h_B x \sim g_A h_B y \sim f_A h'_B x' \sim g_A h'_B y'$. Then $y \succcurlyeq x$ if and only if $y' \succcurlyeq x'$.*

## 4. REPRESENTATIONS WITHOUT PROBABILISTIC SOPHISTICATION

In this section, we provide a representation theorem for preferences that satisfy weak decomposability but do not necessarily satisfy probabilistic

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[13] A measure with this property is sometimes referred to as an atomless measure, but this is confusing since we are concerned with finitely additive measures: in probability theory, an atomless measure refers to a measure $\mu$ for which there is no event $E$ with $\mu(E) > 0$ such that $E' \subset E$ implies either $\mu(E') = 0$ or $= \mu(E)$. An atomless measure is strongly continuous if it is countably additive, but not necessarily so if it is only finitely additive. See Bhaskara Rao & Bhaskara Rao [1].

sophistication or even Savage's weak comparative probability postulate P4. The conditions we give for the sufficiency and necessity parts are not identical, so we break the result into two propositions. In both results, all the conditions except P6 (and the corresponding property 5) can be met when the state space is finite. Indeed, the proofs require minimal modification to cover this case. We chose to retain P6 to keep our framework similar to that of Savage. For this section, we take $\mathscr{X}$ to be a compact interval in the real line, $[\underline{x}, \bar{x}]$, where $\underline{x} < \bar{x}$.[14] Given this setting, we assume outcome continuity, P6**.

*Sufficiency.* The following proposition gives conditions for a representation $V$ that are sufficient for the induced preferences to satisfy weak decomposability.

PROPOSITION 6. *Let $\mathscr{X} = [\underline{x}, \bar{x}]$. For a state space $\mathscr{S}$, a set of events $\mathscr{E}$, and a function $\varphi: \mathscr{X} \times \mathscr{E} \times \mathscr{X} \to \mathbb{R}$, let $\mathscr{N} := \{A \in \mathscr{E} : \varphi(x, A, w) = 0 \text{ for all } x, w \text{ in } \mathscr{X}\}$. Suppose that $\mathscr{S}$, $\mathscr{E}$, and $\varphi$ satisfy the following properties*:

1. *$\varphi$ is continuous in its first and third arguments*;

2. *for all events $A$ in $\mathscr{E} \setminus \mathscr{N}$, $\varphi(., A, .)$ is increasing in the first argument and is decreasing in the third argument*;

3. *for all events $A$ in $\mathscr{E}$, and all $x$ in $\mathscr{X}$, $\varphi(x, A, x) = 0$*;

4. *$\varphi$ is* state additive; *that is, for all pairs of events $A$ and $B$ in $\mathscr{E}$, and all $x$ and $w$ in $\mathscr{X}$, if $A \cap B = \varnothing$ then $\varphi(x, A, w) + \varphi(x, B, w) = \varphi(x, A \cup B, w)$*;

5. *$\varphi$ is* small-event continuous; *that is, for all outcomes $x$ and $w$ in $\mathscr{X}$ with $x \neq w$, and all $\varepsilon > 0$, there exists a finite partition $\mathscr{P} = \{A_k : k = 1, ..., K\}$ of $\mathscr{S}$, $\mathscr{P} \subset \mathscr{E}$, such that $|\varphi(x, A_k, w)| < \varepsilon$ for all $A_k \in \mathscr{P}$*.

*And, for all simple acts $f$ in $\mathscr{F}$, let $V(f)$ be the (unique) implicit solution to*:

$$\sum_{x \in \mathscr{X}} \varphi(x, f^{-1}(x), V(f)) = 0. \tag{1}$$

*Then the relation $\succcurlyeq$ induced by the functional $V$ satisfies* P1, P3, P6, P6** *and weak decomposability.*

To help understand this representation, first notice that we can write the standard subjective expected utility model in this form.

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[14] More generally, given P3, it is enough that the outcome space is rich enough for there to exist a utility function $u$ defined over the set of constant acts (i.e., outcomes), and representing preferences over those acts, whose range is such an interval. For a discussion of related issues in the context of risk, see Grant, Kajii & Polak [14].

EXAMPLE 1.   Set $\mathscr{S} := [0, 1]$ and suppose that

$$\bar{V}(f) = \int_s u(f(s))\,\mu(ds) = \sum_{x \in \mathscr{X}} u(x)\,\mu \circ f^{-1}(x), \tag{2}$$

where $u$ is a continuous increasing utility index and where $\mu$ is a strongly continuous finitely additive probability measure. We can rewrite expression (2) as

$$\sum_{x \in \mathscr{X}} (u(x) - \bar{V}(f))\,\mu \circ f^{-1}(x) = 0.$$

If we now set $\bar{\varphi}(x, E, v) := (u(x) - v)\,\mu(A)$, then this reduces to expression (1). It is easy to check that $\bar{\varphi}$ satisfies the six properties of Proposition 6. In this special case, however, the preferences induced by $\bar{V}$ also satisfy P2 and P4. The following is a more general special case.

EXAMPLE 2.   Set $\mathscr{S} := [0, 1]$ and suppose that $V^*(f)$ is given by the solution, $v$, to

$$\int_{s \in \mathscr{S}} \psi(f(s), s, v)\,ds = 0, \tag{3}$$

where the function $\psi: \mathscr{X} \times s \times \mathscr{X} \to \mathbb{R}$ satisfies three properties analogous to properties 1, 2 and 3 above: $\psi$ is continuous in its first and third arguments, increasing in the first and decreasing in the third, and $\psi(x, s, x) = 0$ for all $s$ and $x$.

The function $V^*$ resembles an implicit linear representation of a betweenness preference relation over lotteries (see, for example, Chew [3]). For each $v$ and $s$, we can think of $\psi(., s, v)$ as assigning a "utility" to each outcome, where that utility depends both on $v$ and on the state $s$ in which the outcome occurs. Then, for fixed $v$, the left side of (3) is the expected "state-dependent utility" of the act $f$ with respect to the Lesbesgue measure.

Just like their betweenness-representation analogues, the three properties of $\psi$ ensure that the solution $v$ exists and is unique, thus the preferences induced by $V^*$ satisfy P1 (order). The third property ensures that, for all constant acts $x$, $V^*(x) = x$. Monotonicity in the first argument thus ensures that the induced preferences satisfy P3 (eventwise monotonicity). Small event continuity (P6) and outcome continuity (P6**) are both immediate in this case, so it only remains to show weak decomposability. Monotonicity in the third argument ensures, as with betweenness representations, $V^*(g) \geqslant V^*(f)$ if and only if $\int_{s \in \mathscr{S}} \psi(g(s), s, V^*(f))\,ds \geqslant 0$. Thus, $V^*(g_A f) \geqslant V^*(f)$ implies $\int_{s \in A} \psi(g(s), s, V^*(f))\,ds \geqslant 0$, and $V^*(f_A g) \geqslant V^*(f)$ implies $\int_{s \in \mathscr{S} \setminus A} \psi(g(s), s, V^*(f))\,ds \geqslant 0$. And, combining these two

implications, we are done. This is essentially the idea of the proof of the proposition.

To see that Example 2 is indeed a special case of the functional form in the proposition, rewrite (3) as

$$\sum_{x \in \mathscr{X}} \int_{s \in f^{-1}(x)} \psi(x, s, V^*(f)) \, ds = 0.$$

If we now set $\varphi^*(x, E, v) := \int_{s \in E} \psi(x, s, v) \, ds$, then this reduces to expression (1). The function $\varphi^*$ inherits properties 1, 2 and 3 from $\psi$, and satisfies state-additivity (property 4) and the analogue of small-event continuity (property 5) by construction.

The functional form in Proposition 6 is more general than $V^*$.[15] In particular, state additivity does not imply the existence of a measure (integral) representation like those in expressions (2) and (3) above. Wakker & Zank [30] consider preferences which satisfy all of Savage's first six axioms except P4. Such preferences need not admit a separation between the "probability" and the "utility" of outcomes. They show that, in general, in the absence of an identifiable probability measure, no integration operation can be defined. Since the class of preferences in Proposition 6 are even less restricted than those of Wakker & Zank, we must make do with the additive form of expression (1).[16]

*Necessity.* The following proposition gives conditions for a representation $V$ that are necessary if the represented preferences satisfy weak decomposability and the same technical conditions as above.[17]

PROPOSITION 7. *Let $\mathscr{X} = [\underline{x}, \bar{x}]$. Assume that $\succcurlyeq$ satisfies* P1, P3, P6 *and* P6**. *Then there exists a function $\varphi: \mathscr{X} \times \mathscr{E} \times (\underline{x}, \bar{x}) \to \mathbb{R}$ with the properties: if $A$ is null then $\varphi(x, A, w) = 0$ for all $x, w$ in $\mathscr{X}$; and*

   1. *$\varphi$ is continuous in its first argument;*

   2. *for all events $A$ in $\mathscr{E} \backslash \mathscr{N}$, and all $w$ in $(\underline{x}, \bar{x})$, $\varphi(., A, w)$ is increasing in the first argument*

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[15] For another example of a function that satisfies properties (1)–(6) consider $\tilde{\varphi}(x, A, v) := \int_{s \in A} [x - v] \, p(ds, v)$.

[16] Even when an integral representation exists, we should not identity the measure over states as the subjective beliefs of the agent. Consider, for example, the functional form (3) above. Let $\mu(\cdot)$ be a probability measure with density $\mu'$. Set $\hat{\psi}(x, s, v) := \psi(x, s, v) \frac{1}{\mu'(s)}$. Then $\int_{s \in \mathscr{S}} \hat{\psi}(f(s), s, v) \, d\mu(s) = 0$ induces the same preferences as expression (3). Since there is no particular reason for choosing $\psi$ over $\hat{\psi}$, this shows that any such $\mu$ could serve as beliefs.

[17] The main difference between the two propositions is that, while it is sufficient for the functions $\varphi$ to be continuous and decreasing in the third argument, we do not know if it is necessary.

3.   *for all events A in $\mathscr{E}$, and all x in $\mathscr{X}$, $\varphi(x, A, x) = 0$;*

4.   *$\varphi$ is state additive*;

5.   *$\varphi$ is small-event continuous with respect to all outcomes x and w in*
$(\underline{x}, \bar{x})$, $x \neq w$;

*such that the relation $\geqslant$ on simple acts (excluding the two constant acts $\underline{x}$ and $\bar{x}$) is represented by a utility function V given by the rule* $\sum_{x \in \{y: f^{-1}(y) \notin \mathscr{N}\}} \varphi(x, f^{-1}(x), V(f)) = 0.$[18]

The idea of the proof is, loosely, as follows. Recall from our discussion of Lemma 5 that we can think of how much we have to "augment" each subact to bring it into an indifference set, as inducing a natural ordering over subacts with respect to that indifference set. Moreover, this ordering respects P2. Thus, using Segal's [25, Theorem 2] result, the ordering has an additively separable representation. We then show that if two subacts agree on the intersection of two events, then the representation (still with respect to a given indifference set) agrees on that intersection up to affine transformations. From there, the proof adapts methods of Chew & Epstein [4] and Wakker & Zank [30]. Notice, however, that unlike the former we do not assume probabilistic sophistication (or even P4), and unlike the latter we do not assume P2.[19]

*Extensions.*   Corollary 4 characterizes weak decomposability given the "conditional weak comparative probability" axiom P4$^C$. Propositions 6 and 7 characterize weak decomposabilty without imposing even the ordinary weak comparative probability axiom P4. It remains an open question how to characterise weak decomposability given P4 (in addition to the other axioms assumed above). Example E in the appendix, however, shows that keeping P4 and the other Savage axioms but substituting weak decomposability for the sure thing principle, P2, is not enough to imply probabilistic sophistication. Indeed, the example shows that weak decomposabilty is consistent with the preferences in the Ellsberg paradox.


## 5. RELATED LITERATURE

Skiadas [27, 28] offers a different interpretation of the Savage story involving the businessman and the property. Loosely speaking,[20] Skiadas's

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[18] Notice that the exclusion of $\underline{x}$ and $\bar{x}$ is without loss of generality since by P3 (eventwise monotonicity) these two acts cannot be certainty equivalents for any other acts.

[19] The representation result shown in Vind [29] can be applied if for any non null event $A$ and any outcomes $x$ and $z$, there are acts $f$ and $g$ such that $x_A f \succ z$ and $z \geqslant x_A g$. Under our assumptions, this condition does not necessarily hold, but some suitable modification of Vind's technique may do. See also Fishburn [12].

[20] One reason this is loose is that Skiadis also does not define outcomes as Savage does.

decision maker is endowed with not just one preference relation over acts, but with a collection of preference relations: one for each event that could occur. These "conditional" preferences need not be those derived by "updating" Savage preferences. In particular, the Skiadas conditional preference relation for some event $E$ (denoted $\geqslant^E$) is not just concerned with sub-acts confined to the outcomes if $E$ occurs, but rather applies to entire acts including the outcomes if $E$ does not occur. In the story, we can think of the businessman's forecasting how he would feel if the Republican wins and he has bought the property and comparing this with how he would feel if the Republican wins and he has not bought the property. The businessman might expect such feelings to be affected by knowing what the outcomes would have been had the Democrat won. Similarly, the businessman can forecast his feelings about the entire acts to buy and not to buy if the Democrat wins.

Suppose the businessman forecasts that, in each event he would be happier if he had bought the property: then he should buy. Skiadas calls this "strict coherence." Like weak decomposability, strict coherence can be thought of as a property that allows decision problems to be considered one event at a time. It turns out that weak decomposability and strict coherence are independent (though not inconsistent) properties.[21] The two approaches place different information requirements on the analyst. On the one hand, weak decomposability involves not just the original two acts under consideration (say, $f$ and $g$), but also the spliced compound-acts (say, $f_E g$ and $g_E f$). Strict coherence involves just the two original acts ($f$ and $g$); and no compound acts. On the other hand, weak decomposability applies within a single preference relation: $\geqslant$. Strict coherence applies across several preference relations: not just the "unconditional" preference relation $\geqslant$, but also those associated with each sub-event under consideration ($\geqslant^E$ and $\geqslant^{S\setminus E}$). Observing these conditional preference relations (other than by introspection) may prove difficult.

## APPENDIX

*Proof of Lemma* 1. Let $b$ denote a best outcome and $w$ denote a worst outcome in the range of acts in question. First, observe that P3 and P6 imply the following continuity property. If $g \succ f$ and $A$ is a non-null event with $g(s) \succ_{\mathscr{X}} w$ for all $s$ in $A$, then there exists a non-null event $E \subset A$ such

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[21] Grant, Kajii and Polak [15] show that a Skiadis-type agent can satisfy strict coherence but violate decomposability in every conditional preference. Alternatively, her conditional preference for each singleton-state event could satisfy decomposability but she could fail to satisfy strict coherence.

that $g \succ w_E g \succ f$. To see this, notice that, by P6, there exists a finite partition $\{E_1, ..., E_n\}$ of $\mathscr{S}$, such that $w_{E_i} g \succ f$ for all $i = 1, ..., n$. As the partition is finite and $A$ is non-null, there exists an element, say $E_1$, that has a non-null intersection with $A$. Set $E = A \cap E_1$. Since, by construction, $g$ is "better" than $w$ on $A$, by P3 (exploiting the fact that $g$ has finite range), $g \succ w_E g \succcurlyeq w_{E_1} g \succ f$ as desired. By a similar argument, if $g \succ f$ and $A'$ is a non-null event with $b \succ_{\mathscr{X}} f(s)$ for all $s$ in $A'$, there exists a non-null event $E' \subset A'$ such that $g \succ b_{E'} f \succ f$.

Weak decomposability implies (1): Suppose $g_A f \succcurlyeq f$ and $f_A g \succcurlyeq f$ but, contrahypothesis, $f \succ g$. If $A$ is null then $g \sim f_A g$, which contradicts $f_A g \succcurlyeq f$ and $f \succ g$. Similarly, if $S \backslash A$ is null then $g \sim g_A f$ which contradicts $g_A f \succcurlyeq f$ and $f \succ g$. So assume both $A$ and $S \backslash A$ are not null. Let $B' := \{s \in A : b \succ_{\mathscr{X}} g(s)\}$. If $B'$ is null then, by P3, $g \succcurlyeq f_A g$, which again contradicts $f_A g \succcurlyeq f$ and $f \succ g$. So assume $B'$ is not null. Since $f \succ g$, applying the observation above (with $B'$ in the role of $A'$), there exists an event $E' \subset B'$ such that $f \succ b_{E'} g$. Similarly, if the set $B'' := \{s \in \mathscr{S} \backslash A : b \succ_{\mathscr{X}} g(s)\}$ is null then, by P3, $g \succcurlyeq g_A f$, which contradicts $g_A f \succcurlyeq f$ and $f \succ g$. So assume that $B''$ is also not null. By construction, $b \succ_{\mathscr{X}} [b_{E'} g](s)$ for all $s$ in the non-null set $B''$ and $f \succ b_{E'} g$. Applying the above observation again (with $B''$ now in the role of $A'$), there exists an event $E'' \subset B''$ such that $f \succ b_{E'} b_{E''} g$. Now, set $g' := b_{E'} b_{E''} g$. By P3, we have $g'_A f = b_{E'} g_{A \backslash E'} f \succ g_A f$, and $f_A g' = f_A b_{E''} g \succ f_A g$. So we get $g'_A f \succ f$ and $f_A g' \succ f$ but $f \succ g'$, a contradiction to weak decomposability.

(1) implies (2):   Suppose $g_A f \succ f$ and $f_A g \succcurlyeq f$ but, contrahypothesis, $f \succcurlyeq g$. Since $g_A f \succ f$, the event $A$ is not null. If $g(s) \sim_{\mathscr{X}} w$ on $A$ except for a null event, then, by P3, $g_A f \succ f$ could not hold. So assume that $g \succ_{\mathscr{X}} w$ on a non-null event in $A$. Applying the observation, find a non-null event $E \subset A$ with $w_E g_{A \backslash E} f \succ f$. Since $f \succcurlyeq g$, by P3, $f \succ w_E g$. But if we set $g' = w_E g$, we get a contradiction to (1).

(2) implies (3):   Suppose $g \succ f_A g$ and $g \succ g_A f$ but, contrahypothesis, $f \succcurlyeq g$. If $f_A g \succcurlyeq g_A f$, we get $f \succcurlyeq g \succ f_A g \succcurlyeq g_A f$. Set $\hat{f} := f_A g$ and $\hat{g} := g_A f$, thus $\hat{f}_A \hat{g} = f$ and $\hat{g}_A \hat{f} = g$. So $\hat{f}_A \hat{g} \succcurlyeq \hat{g}_A \hat{f} \succ \hat{f} \succcurlyeq \hat{g}$, a contradiction to (2). Conversely, if $g_A f \succ f_A g$, we get $f \succcurlyeq g \succ g_A f \succ f_A g$. So, $\hat{f}_A \hat{g} \succcurlyeq \hat{g}_A \hat{f} \succ \hat{g} \succ \hat{f}$, again contradicting (2).

(3) implies (4):   This case is analogous to weak decomposability implies (1).

(4) implies (5):   This case is analogous to (1) implies (2).

(5) implies weak decomposability:   This case is analogous to (2) implies (3).

We have established that weak decomposability is equivalent to (1) through (5), so weak decomposability implies (6) since (6) is an immediate

consequence of (1) and (4). By P6*, if $g \succ f_A g$, then there is an event $E \subset A$ such that $b_E f_{A \setminus E} g \sim g$. So if (6) is satisfied but (3) is violated, i.e., $g \succ f_A g$ and $g \succ g_A f$ but $f \succcurlyeq g$, we can apply this property (substituting $b$ into $f$ first on $A$ and then on $S \setminus A$) to obtain an act, $\hat{f}$, such that $g \sim \hat{f}_A g$ and $g \sim g_A \hat{f}$. But, given P3, by construction, $\hat{f} \succ f \succ g$, contradicting (6). ∎

*Proof of Proposition* 2. To show weak decomposability implies DPS*, fix a finite partition $\mathscr{P} := \{A_1, ..., A_n\}$ and a pair of acts $f$, $g$ in $\mathscr{F}$ with $f \sim g$ and $f \succcurlyeq h$, for all acts $h$ as described in the proposition. We will proceed by induction. Suppose as an induction hypothesis first that $f$ is indifferent to the act in which the first $k-1$ elements of the partition are determined by $g$ and the remaining $n-k+1$ elements of the partition are determined by $f$. For $k = 1$, take the act to be $f$ itself. So, the induction hypothesis holds for $k = 1$. Consider $k > 1$.

Suppose $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succ f_{A_k} g$. Then, by the contrapositive of Lemma 1 part (1) applied to the event $A_k$, either $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succ g_{A_1 \cup \cdots \cup A_{k-1}} f$ which is indifferent to $f$ (by the induction hypothesis); or $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succ g$ which is indifferent to $f$ (by hypothesis of DPS*). But, (by a hypothesis of DPS*), $f \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f$: a contradiction. An analogous argument rules out $f_{A_k} g \succ g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f$. Therefore we have $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \sim f_{A_k} g$.

As $g \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1}} f$, it follows from the contra-positive of Lemma 1 part (3) applied to the event $A_k$ that either $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1}} f$ or $f_{A_k} g \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1}} f$. But, by the conclusion of the last paragraph, this implies $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1}} f$. Combining we have $f \succcurlyeq$ (by hypothesis of DPS*) $g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f \succcurlyeq g_{A_1 \cup \cdots \cup A_{k-1}} f \sim$ (induction hypothesis) $f$. So, $f \sim g_{A_1 \cup \cdots \cup A_{k-1} \cup A_k} f$. But since the ordering of the elements in the partition was arbitrary, we are done.

Next, we will show: not weak decomposability implies not DPS*. Assume weak decomposability does not hold. That is, there exists a pair of acts $f$ and $g$, and an event $A$ such that $g_A f \succ f$, $f_A g \succ f$ and yet $f \succcurlyeq g$. Let $w$ be the worst outcome in the ranges of $f$ and $g$. Applying P6* twice, there exists a (non-null) event $E^* \subseteq A$ such that $w_{E^*} g_{A \setminus E^*} f \sim f$ and there exists a (non-null) event $E^{**} \subseteq S \setminus A$ such that $w_{E^{**}} f_A g \sim f$. Consider the four acts of the form $h = h_A^1 h^2$ where $h^i \in \{w_{E^*} g_{A \setminus E^*} f, w_{E^{**}} f_A g\}$. By construction, we have $w_{E^*} g_{A \setminus E^*} f \sim w_{E^{**}} f_A g$ and $w_{E^*} g_{A \setminus E^*} f \sim f$. The fourth such act $w_{E^*} w_{E^{**}} g \prec g$, by P3 (this relation must be strict given $g_A f \succ w_{E^*} g_{A \setminus E^*} f$). Thus, given $f \succcurlyeq g$, we have $w_{E^*} g_{A \setminus E^*} f \succ w_{E^*} w_{E^{**}} g$, violating DPS*. ∎

*Proof of Proposition* 3. We note that under the maintained assumptions, the associated measure $\mu$ is strongly continuous; that is, for any $\varepsilon > 0$, there exists a finite partition $\{E_1, ..., E_K\}$ of $\mathscr{S}$ such that $\mu(E_k) < \varepsilon$ for

every $k$. It is known that the range of a strongly continuous measure is convex, hence in particular, there is an event with probability $\frac{1}{2}$.

LEMMA A. *Let $V$ be a continuous function on the set of lotteries $\mathscr{L}(\mathscr{X})$. If $V$ satisfies the property that, for every pair of lotteries $P$ and $Q$, $V(P) = V(\frac{1}{2}P + \frac{1}{2}Q)$ implies $V(P) \leqslant (respectively, \geqslant) V(Q)$ then $V$ is quasi-convex (resp. quasi-concave).*

*Proof.* Suppose on the contrary that $V$ is not quasi-convex. Then there exist lotteries $\bar{P}$ and $\bar{Q}$ and a weight $\alpha$ in $(0, 1)$, such that $V(\bar{P}) = V(\bar{Q})$ but $V(\bar{P}) < V(\alpha\bar{P} + (1 - \alpha)\bar{Q})$. Define a function $v$ on $[0, 1]$ by the rule $v(x) = V(x\bar{P} + (1 - x)\bar{Q})$. By construction, $v$ is continuous, so it attains its maximum value $\bar{v}$ on $[0, 1]$. Let $w := \sup\{x \in [0, 1] : v(x) = \bar{v}\}$. By construction, $v(w) = \bar{v} > v(1)$. We say that $v(w)$ is a strict (respectively, weak) local maximum if there is a $\bar{\delta} > 0$ such that, for all $x$ in $(w - \bar{\delta}, w + \bar{\delta})$, $x \neq w$, $v(x) < v(w)$ (respectively, $\leqslant$). Similarly define strict and weak local minima.

Suppose $v(w)$ is a strict local maximum. Then, by continuity, there is a $\delta > 0$, such that no $x$ in $(w - 4\delta, w + 4\delta)$ is a weak local minimum. Therefore, if $y$ is in $(w - 4\delta, w)$ then for all $x$ in $(y, w)$, we have $v(y) < v(x) < v(w)$; and if $y$ is in $(w, w + 4\delta)$ then for all $x$ in $(w, y)$, we have $v(y) < v(x) < v(w)$. Without loss of generality, let $v(w - \delta) \leqslant v(w + \delta)$. Let $\hat{x} \in [w - \delta, w)$ be such that $v(\hat{x}) = v(w + \delta)$. Let $t = w + \delta - \hat{x}$. Set $\hat{y} := (w + \delta) + t$. By construction $\hat{y}$ is in $(w, w + 4\delta)$, so $v(\hat{y}) < v(w + \delta)$. But, $w + \delta = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$: a contradiction.

Suppose, then, that $v(w)$ is not a strict local maximum. By construction, $w < 1$, and $v(x) < v(w)$ for all $x$ in $(w, 1]$. Let $2\delta = 1 - w$. Since $v(w)$ is not a strict local maximum, there exists an $\hat{x}$ in $(w - \delta, w)$ such that $v(\hat{x}) = v(w)$. Set $\hat{y} := w + (w - \hat{x})$. By construction, $\hat{y} < 1$, $v(\hat{y}) < v(w)$, and $w = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$: a contradiction. The argument for quasi-concavity is similar.  ∎

We can now proceed to prove the proposition.

(i) $\Rightarrow$ (ii)  Suppose that (ii) fails to hold. Then from the lemma there exists lotteries $P$ and $Q$ for which $V(P) = V(\frac{1}{2}P + \frac{1}{2}Q) > V(Q)$ or $V(P) > V(\frac{1}{2}P + \frac{1}{2}Q) = V(Q)$. Since the argument is symmetric, assume the former. As $\mu$ is non-atomic there exists an event $A$ with $\mu(A) = 1/2$ and there exist two acts $\hat{f}$ and $\hat{g}$ which satisfy: $\mu \circ \hat{f}^{-1}(x) = \mu(\hat{f}^{-1}(x) \cap A)/\mu(A) = P(x)$ and $\mu \circ \hat{g}^{-1}(x) = \mu(\hat{g}^{-1}(x) \cap A)/\mu(A) = Q(x)$ for all $x$ in $f(\mathscr{S}) \cup g(\mathscr{S})$. So by probabilistic sophistication $\hat{f} \sim \hat{f}_A\hat{g} \sim \hat{g}_A\hat{f} \succ \hat{g}$. But by setting $f := \hat{f}$ and $g := \hat{g}$ we have a violation of implication (6) of Lemma 1.

(ii) $\Rightarrow$ (i)  To show weak decomposability we require $g_A f \succ f$ and $f_A g \succ f$ to imply $g \succ f$. Notice first that neither $A$ nor $\mathscr{S}\backslash A$ may be null.

For $E$ in $\{A, \mathscr{S}\backslash A\}$ let $P_E, Q_E$ in $\mathscr{L}_0(\mathscr{X})$ denote the lotteries defined by the rule: $P_E(x) := \mu(f^{-1}(x) \cap E)/\mu(E)$ and $Q_E(x) := \mu(g^{-1}(x) \cap E)/\mu(E)$. Thus, $g_A f \succ f$ corresponds to $V(\mu(A)\, Q_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A}) > V(\mu(A)\, P_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A})$ and $f_A g \succ f$ to $V(\mu(A)\, P_A + (1 - \mu(A))\, Q_{\mathscr{S}\backslash A}) > V(\mu(A)\, P_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A})$. As $V$ is quasi-concave in probability mixtures it follows that $V(\frac{1}{2}[\mu(A)\, P_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A}] + \frac{1}{2}[\mu(A)\, Q_A + (1 - \mu(A))\, Q_{\mathscr{S}\backslash A}]) > V(\mu(A)\, P_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A})$. Applying the quasi-concavity again yields $V(\mu(A)\, Q_A + (1 - \mu(A))\, Q_{\mathscr{S}\backslash A}) > V(\mu(A)\, P_A + (1 - \mu(A))\, P_{\mathscr{S}\backslash A})$. This in turn implies by probabilistic sophistication that $g \succ f$, as required. ∎

*Proof of Lemma* 5. Since $y \succcurlyeq x$, $f_A h_B x \sim g_A h_B y$ implies (by P3) $f_A h_B x \succcurlyeq g_A h_B x$. By hypothesis, $f_A h_B x \succcurlyeq f_A h'_B x'$. These two facts imply (by Lemma 1 part 1) $f_A h_B x \succcurlyeq g_A h'_B x'$. By hypothesis, $f_A h_B x \sim g_A h'_B y'$, hence $g_A h'_B y' \succcurlyeq g_A h'_B x'$. So, by P3, $y' \succcurlyeq x'$. ∎

*Proof of Proposition* 6. We first show that the utility function $V$ is uniquely defined, and therefore that $\succcurlyeq$ satisfies P1. For any act $f$ in $\mathscr{F}$, let $\Phi_f(w) := \sum_x \varphi(x, f^{-1}(x), w)$. By properties 2 and 3, for all events $A$ in $\mathscr{E}\backslash\mathscr{N}$, $\varphi(x, A, w) \geqslant \varphi(x, A, x) = 0$ if and only if $x \succcurlyeq w$. By our choice of $\mathscr{X}$, $\underline{x} \leqslant x \leqslant \bar{x}$ for all $x$ in $f(\mathscr{S})$. Therefore, $\Phi_f(\bar{x}) \leqslant 0 \leqslant \Phi_f(\underline{x})$. So, by property 1 and 2, there exists a unique $w$ ( $=: V(f)$) with $\Phi_f(w) = 0$. For any constant act $x$, we have $V(x) = x$, hence property 2 implies that $\succcurlyeq$ satisfies P3.

We next show that, if $\mathscr{S}$, $\mathscr{E}$, and $\varphi$ satisfy properties 2, 3 and 4, then $\varphi$ is state monotonic; that is, for all pairs of events $A$ and $B$ in $\mathscr{E}$, and all $x$ and $w$ in $\mathscr{X}$, if $A \subset B$ then $|\varphi(x, A, w)| \leqslant |\varphi(x, B, w)|$. To see this, fix $A \subset B$ and fix $x$ and $w$. By the definition of $\mathscr{N}$ and property 3, if $x = w$ then $\varphi(x, A, w) = \varphi(x, B, w) = 0$. By the definition of $\mathscr{N}$ and properties 2 and 3, if $x > w$ then $\varphi(x, E, w) \geqslant 0$ for all events $E$ in $\mathscr{E}$ (with strict inequality for $E$ in $\mathscr{E}\backslash\mathscr{N}$). So, in particular, $\varphi(x, A, w) \geqslant 0$ and $\varphi(x, B\backslash A, w) \geqslant 0$. Property 4 then gives $\varphi(x, A, w) \leqslant \varphi(x, B, w)$. The case for $x < w$ is similar. Now, for P6, fix two acts $f$ and $g$ with $f \succ g$, and an outcome $\hat{x}$ in $\mathscr{X}$. Notice that $f \succ g$ implies that there exists an $\varepsilon > 0$, and a utility level $w$ in $[\underline{x}, \bar{x}]$, for which $\Phi_f(w) > \varepsilon$ and $-\varepsilon > \Phi_g(w)$. Let T be the union of the range of $f$ and $g$, which is a finite set. For each $x \in T \cup \{\hat{x}\}$, use property 5 to find a finite partition $\mathscr{P}_x$ such that $|\varphi(x, A, w)| < \frac{1}{1 + \#(T \cup \{\hat{x}\})} \varepsilon$ for any $A \in \mathscr{P}_x$. Let $\mathscr{P}$ be the coarsest common refinement of $\{\mathscr{P}_x : x \in T \cup \{\hat{x}\}\}$. $\mathscr{P}$ is a finite partition. Pick any $A$ in $\mathscr{P}$. By property 4, $\Phi_f(w) - \Phi_{x_A f}(w) = (\sum_{\{x: A \cap f^{-1}(x) \neq \varnothing\}} \varphi(x, f^{-1}(x) \cap A, w)) - \varphi(\hat{x}, A, w)$. By state monotonicity, $\varphi(x, f^{-1}(x) \cap A, w) < \frac{1}{1 + \#(T \cup \{\hat{x}\})} \varepsilon$ for each $x$ with $A \cap f^{-1}(x) \neq \varnothing$. Since the set $\{x : A \cap f^{-1}(x) \neq \varnothing\}$ has at most $\#(T \cup \{\hat{x}\})$ elements, we have $|\varphi_f(w) - \varphi_{x_A f}(w)| < \varepsilon$. Similarity, $|\varphi_g(w) - \varphi_{x_A g}(w)| < \varepsilon$ for each $A$ in $\mathscr{P}$. These in turn imply $\varphi_{x_A f}(w) > 0$ and $0 > \varphi_{x_A g}(w)$. Thus $x_A f \succ g$ and

$f \succ x_A g$ as required. Outcome continuity (P6**) follows immediately from property 1 (continuity with respect to the first and third arguments).

Finally, for weak decomposability, suppose $g_A f \succ f$ and $f_A g \succ f$. Then, $\sum_x \varphi(x, g^{-1}(x) \cap A, V(f)) + \sum_x \varphi(x, f^{-1}(x) \backslash A, V(f)) > 0$ and $\sum_x \varphi(x, f^{-1}(x) \cap A, V(f)) + \sum_x \varphi(x, g^{-1}(x) \backslash A, V(f)) > 0$. So $\sum_x \varphi(x, g^{-1}(x) \cap A, V(f)) + \sum_x \varphi(x, g^{-1}(x) \backslash A, V(f)) > 0$, which implies $g \succ f$. ∎

*Proof of Proposition* 7.  Parts of the argument adapt ideas of Chew & Epstein [4] and Wakker & Zank [30]. We use the following standard result.

*Fact* 1.  Let $X = \Pi_{i=k}^K X_k$ with $K > 1$ and where $X_k$ is an open interval in $\mathbb{R}$ for every $k$. Suppose $U$ and $V$ are continuous, additively separable on $X$, i.e., $U(\mathbf{x}) = \sum_{k=1}^K u_k(x_k)$ and $V(\mathbf{x}) = \sum_{k=1}^K v_k(x_k)$, and $U(\mathbf{x}) \geqslant U(\mathbf{y})$ if and only if $V(\mathbf{x}) \geqslant V(\mathbf{y})$. Then there is a unique set of constants $a, b_1, ..., b_K$ with $a > 0$, such that $u_k = a v_k + b_k$ for all $k$.

Fix a non null partition $\mathscr{A} = \{A_k : k = 1, ..., K\}$ with $K > 3$, which exists by P6. Each act in $\mathscr{F}^{\mathscr{A}}$ can be regarded as an element of $\mathbb{R}^K$, so we naturally write an act $f$ as $\mathbf{x} := (x_1, ..., x_K)$ where $x_k = f(A_k)$. We use the convention of writing $f_{-k}$ or $\mathbf{x}_{-k}$ for the vector (act) that is obtained by dropping the $k$th element of $f$, and we write $(\mathbf{x}_{-k}, a)$ for the vector where $k$th coordinate is $a$ and the other elements are given by the corresponding elements of vector $\mathbf{x}$. Similarly, we write $(\mathbf{x}_{-(k, k')}, a, b)$ for the vector where $k$th and $k'$th coordinates are replaced with $a$ and $b$, respectively.

For each outcome $z$ in $(\underline{x}, \bar{x})$, and for each coordinate $k$, define $\mathscr{I}(z) = \{f \in \mathscr{F}^{\mathscr{A}} : f \sim z\}$, and $\mathscr{I}_{-k}(z) = \{\mathbf{x}_{-k} \in \mathbb{R}^{K-1} : \exists f \in \mathscr{I}(z), f_{-k} = \mathbf{x}_{-k}\}$. By construction, $(z, ..., z) \in \mathscr{I}_{-k}(z)$, so $\mathscr{I}_{-k}(z)$ is a non-empty set. By continuity, $\mathscr{I}_{-k}(z)$ is a closed subset of $[\underline{x}, \bar{x}]^{K-1}$ hence it is compact. Since $z \in (\underline{x}, \bar{x})$, again by continuity, the set $\mathscr{I}_{-k}(z)$ has a non-empty interior. It can be readily checked that $\mathscr{I}_{-k}(z)$ is the closure of its interior, where the boundary points lie in sets of the form $[\mathscr{I}_{-k}]_i (z) := \{\mathbf{x}_{-k} \in \mathscr{I}_{-k}(z) : \mathbf{y}_{-k} \in \mathscr{I}_{-k}(z) \Rightarrow (\mathbf{y}_{-k})_i \geqslant (\mathbf{x}_{-k})_i\}$, and $[\mathscr{I}_{-k}]^i (z) = \{\mathbf{x}_{-k} \in \mathscr{I}_{-k}(z) : \mathbf{y}_{-k} \in \mathscr{I}_{-k}(z) \Rightarrow (\mathbf{y}_{-k})_i \leqslant (\mathbf{x}_{-k})_i\}$, where $(\mathbf{y}_{-k})_i$ and $(\mathbf{x}_{-k})_i$ are the $i$th element of $\mathbf{y}_{-k}$ and $\mathbf{x}_{-k}$, respectively.

Let $c_k(.; z): \mathscr{I}_{-k}(z) \to \mathscr{X}$ be defined by the rule:

$$(\mathbf{x}_{-k}, c_k(\mathbf{x}_{-k}; z)) \sim z.$$

The function $c_k(\mathbf{x}_{-k}; z)$ is well defined by P1 (order), P3 (eventwise monotonicity), continuity and the construction of $\mathscr{I}_{-k}(z)$. Define a binary relation $\succcurlyeq_k^z$ on $\mathscr{I}_{-k}(z)$ by the rule:

$$\mathbf{x}_{-k} \succcurlyeq_k^z \mathbf{y}_{-k} \Leftrightarrow c_k(\mathbf{x}_{-k}; z) \leqslant c_k(\mathbf{y}_{-k}; z).$$

So $\succcurlyeq_k^z$ is a well defined continuous relation on $\mathscr{I}_{-k}(z)$. Since each $A_k$ in $\mathscr{A}$ is non null, this preference relation is strictly monotonic by P3. From Lemma 5 in Section 3, weak decomposability implies that each such relation $\succcurlyeq_k^z$ satisfies Debreu [6]'s separability condition. We claim that each $\succcurlyeq_k^z$ admits a continuous, additively separable utility representation $U^k(\cdot; z)$ on $\mathscr{I}_{-k}(z)$ by Segal [25, Theorem 1 and 2]. We shall argue that all the conditions of his theorems are satisfied. The relation $\succcurlyeq_k^z$ is continuous and strictly monotone.

We have seen that the domain $\mathscr{I}_{-k}(z)$ is compact and equals the closure of its interior. The "richness of boundary" condition (the fifth condition in Theorem 2) is satisfied since, by continuity, each $[\mathscr{I}_{-k}]_i(z)$ and $[\mathscr{I}_{-k}]^i(z)$ has a non-empty relative interior.

Since the relation $\succcurlyeq$ is continuous and strictly monotonic on $\mathscr{X}^K$, each of its indifference surfaces is a connected (in fact arc-connected) subset of $\mathbb{R}^K$. Thus the domain $\mathscr{I}_{-k}(z)$ is connected since it is the projected image of a connected set. To show that each truncation of $\mathscr{I}_{-k}(z)$ (Segal's $S(i, c)$) and each indifferent surface of $\succcurlyeq_k^z$ is connected, we can apply the argument of the Lemma in Chew, Epstein & Wakker [5, p. 184].

For each $k$ write $U^k(\mathbf{x}_{-k}; z) = \sum_{i \neq k} u_i^k(x_i; z)$, where, by construction and P3, for each $k$, $u_i^k$ is increasing and continuous in $x_i$. We can normalize each $u_i^k(z; z) = 0$ for every $i \neq k$. By construction, for each $k$ and $k'$ with $k \neq k'$, and each $i \notin \{k, k'\}$, the domains of the functions $u_i^k(\cdot; z)$ and $u_i^{k'}(\cdot; z)$ are the same. The result below shows that these functions are essentially the same.

LEMMA B. *For any $k$ and $k'$ with $k \neq k'$, and for any $i \notin \{k, k'\}$, we can find a unique positive constant $\beta_{k'}^k$ such that $\beta_{k'}^k u_i^k(\cdot; z) = u_i^{k'}(\cdot; z)$. Moreover, if $k$, $l$, and $m$ are distinct, then $\beta_l^k \beta_m^l = \beta_m^k$.*

*Proof.* Without loss of generality, set $k := K-1$, $k' := K$. Consider the set $\mathscr{I}_{-(k, k')}(z) = \{\mathbf{x} \in \mathbb{R}^{K-2} : \exists a \in \mathscr{X}, \text{ such that } (\mathbf{x}, a, a) \sim z\}$. By continuity this set has a non-empty interior. For each $i \notin \{K-1, K\}$, if $x_i$ is in the domain of $u_i^K(\cdot; z)$ then there exists a vector $\mathbf{x}$ in $\mathbb{R}^{K-2}$ such that $(\mathbf{x}_{-i}, x_i) \in \mathscr{I}_{-(k, k')}(z)$. Similarly, for each $x_i$ in the domain of $u_i^{K-1}(\cdot; z)$. We shall show that the functions $\sum_{i \neq k, k'} u_i^k(x_i; z)$ and $\sum_{i \neq k, k'} u_i^{k'}(x_i; z)$ induce the same preordering on $\mathscr{I}_{-(k, k')}(z)$. For any $\mathbf{x}_{-(K, K-1)}$, $\mathbf{y}_{-(K, K-1)}$ in $\mathscr{I}_{-(k, k')}(z)$, there are outcomes $a$ and $b$ such that $(\mathbf{x}_{-(K, K-1)}, a, a) \sim (\mathbf{y}_{-(K, K-1)}, b, b) \sim z$. Thus by construction, for any $\mathbf{x}_{-(K, K-1)}$, $\mathbf{y}_{-(K, K-1)}$, we have: $\sum_{i \neq k, k'} u_i^k(x_i; z) \geqslant \sum_{i \neq k, k'} u_i^k(y_i; z) \Leftrightarrow U^k((\mathbf{x}_{-(K, K-1)}, a, a)_{-k}; z) \geqslant U^k((\mathbf{y}_{-(K, K-1)}, a, a)_{-k}; z) \Leftrightarrow a \leqslant b$ (by the construction of $\succcurlyeq_k^z$) $\Leftrightarrow U^{k'}((\mathbf{x}_{-(K, K-1)}, a, a)_{-k'}; z) \geqslant U^{k'}((\mathbf{y}_{-(K, K-1)}, a, a)_{-k'}; z) \Leftrightarrow \sum_{i \neq k, k'} u_i^{k'}(x_i; z) \geqslant \sum_{i \neq k, k'} u_i^{k'}(y_i; z)$. By Fact 1, with our normalization $u_i^k(z; z) = 0$, for any $k$ and $k'$ with $k \neq k'$, we can find a unique positive constant $\beta_{k'}^k$ such that

$\beta^k_{k'} u^k_i(\cdot; z) = u^{k'}_i(\cdot; z)$ for any $i \notin \{k, k'\}$. To see the second half of the claim, since $K \geqslant 4$, pick an $i$ that is distinct from $k$, $l$, or $m$. We have $u^m_i(\cdot; z) = \beta^l_m(u^l_i(\cdot; z)) = \beta^l_m(\beta^k_l u^k_i(\cdot; z))$, and $u^m_i(\cdot; z) = \beta^k_m(u^k_i(\cdot; z))$. So $\beta^k_l \beta^l_m = \beta^k_m$, as required. ∎

Now construct "utility" functions $\xi_k$, $k = 1, ..., K$, by the rule: $\xi_1(\cdot; z) = \beta^K_1 u^K_1(\cdot; z)$, and $\xi_i(\cdot; z) = u^1_i(\cdot; z)$ for $i > 1$. Write $\Xi(\mathbf{x}; z) = \sum_k \xi_k(x_k; z)$.

LEMMA C. *If $\mathbf{x}$ and $\mathbf{y}$, both in $\mathbb{R}^K$, have a common component and $\mathbf{x} \sim \mathbf{y} \sim z$, then $\Xi(\mathbf{x}; z) = \Xi(\mathbf{y}; z)$.*

*Proof.* Let $\mathbf{x} \sim \mathbf{y} \sim z$ and suppose we have $\Xi(\mathbf{x}; z) > \Xi(\mathbf{y}; z)$ while $\xi_k(x_k; z) = \xi_k(y_k; z)$ for some $k$. Consider two cases. If $\xi_1(x_1; z) = \xi_1(y_1; z)$, then $\Xi(\mathbf{x}; z) > \Xi(\mathbf{y}; z)$ implies $u^1_2(x_2; z) + \cdots + u^1_K(x_K; z) > u^1_2(y_2; z) + \cdots + u^1_K(y_K; z)$, but then $c_1(\mathbf{x}_{-1}; z) < c_1(\mathbf{y}_{-1}; z)$ must hold by the construction of $U^1$. But, since $\mathbf{x} \sim \mathbf{y} \sim z$, $c_1(\mathbf{x}_{-1}; z) = x_1$ and $c_1(\mathbf{y}_{-1}; z) = y_1$: a contradiction. In the second case $\xi_k(x_k; z) = \xi_k(y_k; z)$ for some $k > 1$. Then, by the definition of $\xi_i$, $\beta^1_k \sum_{i \neq k} \xi_i(x_i; z) = \beta^1_k \beta^K_1 u^K_1(x_1; z) + \sum_{i \neq 1, k} \beta^1_k u^1_i(x_i; z) = \sum_{i \neq k} u^k_i(x_i; z)$. So, $\Xi(\mathbf{x}; z) > \Xi(\mathbf{y}; z)$ implies $\sum_{i \neq k} u^k_i(x_i; z) > \sum_{i \neq k} u^k_i(y_i; z)$, which implies $c_k(\mathbf{x}_{-k}; z) < c_k(\mathbf{y}_{-k}; z)$. But, since $\mathbf{x} \sim \mathbf{y} \sim z$, $c_k(\mathbf{x}_{-k}; z) = x_k$ and $c_k(\mathbf{y}_{-k}; z) = y_k$: a contradiction. ∎

In the proof of the Lemma, for fixed $z$, we showed that for all $k$, $\beta^1_k \sum_{i \neq k} \xi_i(x_i; z) = \sum_{i \neq k} u^k_i(x_i; z)$ (where $\beta^1_1 = 1$). We use this trick again below.

LEMMA D. *For any $\mathbf{x}$ in $\mathbb{R}^K$, $\mathbf{x} \sim z$ holds if and only if $\Xi(\mathbf{x}; z) = 0$.*

*Proof.* It suffices to show that $(x_1, ..., x_K) \sim z$ if and only if $\Xi(\mathbf{x}; z) = 0$. Suppose $\mathbf{x} = (x_1, ..., x_K) \sim z$, but $\Xi(\mathbf{x}; z) > 0$. We first claim that $\xi_k(x_k; z) > 0$ for every $k$ is impossible. To see this, similar to before, $\sum^K_{k=2} \xi_k(x_k; z) > 0$ implies $c_1(\mathbf{x}_{-1}; z) < z$. But since $\mathbf{x} \sim z$, we have $c_1(\mathbf{x}_{-1}; z) = x_1$. So, by monotonicity, $\xi_1(x_1; z) < \xi_1(z; z) = 0$: a contradiction. The same argument rules out $\xi_k(x_k; z) < 0$ for every $k$. So, we can assume $\xi_k(x_k; z) > 0$ and $\xi_{k'}(x_{k'}; z) < 0$ for some $k$ and $k'$.

Now $\xi_k(x_k; z) > 0$ implies $x_k > z$. If $\sum_{i \neq k} \xi_i(x_i; z) \geqslant 0$, then, by the same reasoning as the previous lemma, we get $\sum_{i \neq k} u^k_i(x_i; z) \geqslant \sum_{i \neq k} u^k_i(z_i; z)$, so $\mathbf{x}_{-k} \succsim^z_k z_{-k}$. But, since $\mathbf{x} \sim z$, this implies $x_k \leqslant z$: a contradiction. So $\sum_{i \neq k} \xi_i(x_i; z) < 0$ must hold. Hence, we have $\Xi(\mathbf{x}; z) = \Xi((\mathbf{x}_{-(k, k')}, x_k, x_{k'}); z) > 0 > \sum_{i \neq k} \xi_i(x_i; z) + \xi_k(z; z) = \Xi((\mathbf{x}_{-(k, k')}, z, x_{k'}), z)$.

We claim that there is an outcome $a^1$ with $\xi_k(x_k; z) > \xi_k(a^1; z) > \xi_k(z; z) = 0$ and such that $\Xi((\mathbf{x}_{-k}, a^1); z) = 0$. To see this, note

by construction that for all $y_k \in (z, x_k)$, $(\mathbf{x}_{-(k, k')}, z, x_{k'}) \prec (\mathbf{x}_{-(k, k')}, y_k, x_{k'})$ $\prec z \precsim (\mathbf{x}_{-(k, k')}, x_k, z)$. Hence, by the continuity of $\succsim$, there exists $y_{k'} \in (x_{k'}, z)$ such that $(\mathbf{x}_{-(k, k')}, y_k, y_{k'}) \sim z$. Therefore the domain of $\xi_k(\,\cdot\,; z)$ contains $[z, x_k]$. Since $\xi_k(\,\cdot\,; z)$ is continuous, the claim follows by the intermediate value theorem. Set $\mathbf{w}^1 := (\mathbf{x}_{-k}, a^1)$. By construction, $x_k > a^1 > z$, and so $\mathbf{w}^1 \prec \mathbf{x} \sim z$.

We claim there exists $b^1 \in (x_{k'}, z]$, such that $(\mathbf{x}_{-(k, k')}, a^1, b^1) \sim z$. To establish this, it is enough to show $(\mathbf{x}_{-(k, k')}, a^1, z) \succsim z$. Suppose on the contrary, $z \succ (\mathbf{x}_{-(k, k')}, a^1, z)$. Then there exists $a \in (a^1, x_k]$ such that $(\mathbf{x}_{-(k, k')}, a, z) \sim z$, since $(\mathbf{x}_{-(k, k')}, x_k, z) \succsim z$. Furthermore, since $\Xi((\mathbf{x}_{-k}, a); z) = 0$ so $\Xi((\mathbf{x}_{-(k, k')}, a, z); z) > 0$, that is, $\sum_{i \neq k, k'} \xi_i(x_i; z) + \xi_k(a; z) > 0$. Hence, we have $c_{k'}((\mathbf{x}_{-(k, k')}, a); z) < z$, which contradicts $(\mathbf{x}_{-\{k, k'\}}, a, z) \sim z$.

To summarize: We started with $\mathbf{x} \sim z$, but $\Xi(\mathbf{x}; z) > 0$. We have $x_k > a^1 > z \geqslant b^1 > x_{k'}$; and we have constructed a vector $\mathbf{w}^1 := (\mathbf{x}_{-(k, k')}, a^1, x_{k'}) \prec z$ but with $\Xi(\mathbf{w}^1; z) = 0$, and a vector $\mathbf{x}^1 = (\mathbf{x}_{-(k, k')}, a^1, b^1) \sim z$, but (like the original $\mathbf{x}$) with $\Xi(\mathbf{x}^1; z) > 0$. So by the same construction, we can find an $a^2$ with $a^1 > a^2 > z$, and a $b^2$ with $z \geqslant b^2 > b^1$. So if $b^1 = z$, we would have obtained a contradiction. As before, let $\mathbf{w}^2 := (\mathbf{x}_{-(k, k')}, a^2, b^1) \prec z$ but with $\Xi(\mathbf{w}^2; z) = 0$, and $\mathbf{x}^2 := (\mathbf{x}_{-(k, k')}, a^2, b^2) \sim z$ but with $\Xi(\mathbf{x}^2; z) > 0$. Repeating this process, we obtain sequences $\{\mathbf{x}^n : n = 1, ...\}$ and $\{\mathbf{w}^n : n = 1, ...\}$, where $\Xi(\mathbf{w}^n; z) = 0$ and $\mathbf{x}^n \sim z$ for all $n$, and their $k$th and $k'$th components $a^n$ and $b^n$ constitute monotone bounded sequences. Thus both sequences converge, and $\lim \mathbf{x}^n$ and $\lim \mathbf{w}^n$ must be the same by construction. Let $\bar{\mathbf{x}}$ be the common limit point. By continuity, $\lim \mathbf{x}^n = \bar{\mathbf{x}} \sim z \sim \mathbf{x}$, and, by the continuity of $\Xi$, $\lim \Xi(\mathbf{w}^n; z) = \Xi(\bar{\mathbf{x}}; z) = 0$. Since the $K - 2$ unchanged components of $\bar{\mathbf{x}}$ are equal to those of $\mathbf{x}$, we must have $\Xi(\bar{\mathbf{x}}; z) = \Xi(\mathbf{x}; z)$ by the previous lemma, but this contradicts $\Xi(\mathbf{x}; z) > 0$. An analogous argument shows that it is also impossible to have $\mathbf{x} \sim z$ and $\Xi(\mathbf{x}; z) < 0$. So $x \sim z$ implies $\Xi(\mathbf{x}; z) = 0$.

Conversely, suppose $\Xi(\mathbf{x}; z) = 0$ but $z \succ \mathbf{x}$. Then we can start with $\mathbf{w}^1$ in the construction of the sequences used above to obtain a contradiction. Similarly, $\Xi(\mathbf{x}; z) = 0$ but $\mathbf{x} \succ z$ is also impossible. ∎

We have shown that $\Xi(\mathbf{x}; z) = 0$ holds if and only if $\mathbf{x} \sim z$. For each $\mathbf{x}$ in $\mathbb{R}^K$, define $V(\mathbf{x}) = z$ such that $z$ is the (unique) outcome indifferent to $\mathbf{x}$. The function $V$ is continuous, and is given by the rule $\Xi(\mathbf{x}, V(\mathbf{x})) \equiv \sum_k \xi_k(x_k, V(\mathbf{x})) = 0$.

Now we are ready to construct $\varphi$. Fix a non-null partition $\mathscr{A}$ with four elements as a reference point, and construct the function $\Xi^{\mathscr{A}}$ that is associated with $\mathscr{A}$ as above. With slight abuse of notation, write $\Xi^{\mathscr{A}}(f; z)$ for $\Xi^{\mathscr{A}}(\mathbf{x}; z)$ where $\mathbf{x}$ is the vector associated with the act $f$ measurable with respect to $\mathscr{A}$. For each non-null event $E$, let $\mathscr{A}(E)$ be the partition

generated by $\mathscr{A}$ and $\{E\}$, and construct $\varXi^{\mathscr{A}(E)}$ as above for $\mathscr{A}(E)$. By construction, $\varXi^{\mathscr{A}(E)}$ induces the same preference relation as $\varXi^{\mathscr{A}}$ on acts measurable with respect to $\mathscr{A}$. So, Fact 1, we can normalize $\varXi^{\mathscr{A}(E)}$ by a unique positive scalar for each $z$ in such a way that $\varXi^{\mathscr{A}(E)}(f; z) = \varXi^{\mathscr{A}}(f, z)$ for any $f$ that is measurable with respect to $\mathscr{A}$. So, by setting $\varphi(x, E, z) = \varXi^{\mathscr{A}(E)}(x_E z; z)$ if $E$ is non-null, and $\varphi(x, E, z) = 0$ otherwise, we have constructed a well-defined function on $\mathscr{X} \times \mathscr{E} \times \mathbb{R}$.

To see $\varphi$ has the desired properties, take any simple act $f$ and let $\mathscr{B} := \{E_1, ..., E_K\}$ be the coarsest partition of $S$ containing $\mathscr{A}$, for which $f$ is measurable. Construct $\varXi^{\mathscr{B}}$ and normalize it as above. Then, for any $E_i$ in $\mathscr{B}$, by construction, $\varXi^{\mathscr{B}}(x_{E_i} z; z) = \varXi^{\mathscr{A}(E_i)}(x_{E_i} z; z)$, so $\varphi(x, E, z) = \varXi^{\mathscr{B}}(x_{E_i} z; z)$. Notice that $\varXi^{\mathscr{B}}(x_{E_i} z; z) = \xi_i^{\mathscr{B}}(x, z)$ if $E_i$ is non-null. Therefore, $\sum_k \varphi(f(E_k), E_k, z) = 0$ if and only if $\sum_k \xi_k^{\mathscr{B}}(f(E_k); z) = 0$ which, by construction, is equivalent to $f \sim z$. Moreover $\varphi(x, E_i \cup E_j, z) = \xi_i^{\mathscr{B}}(x, z) + \xi_j^{\mathscr{B}}(x, z) = \varphi(x, E_i, z) + \varphi(x, E_j, z)$, so $\varphi$ is state additive.

To see that $\varphi$ satisfies small event-continuity, fix $x$ and $w$ in $(\underline{x}, \bar{x})$ and take $x > w$ (the case for $x < w$ is similar). Suppose $|\varphi(\underline{x}, \mathscr{S}, w)| \leqslant |\varphi(\bar{x}, \mathscr{S}, w)|$ (an analogous argument holds when the inequality is reversed). Fix an $\varepsilon < \varphi(\underline{x}, \mathscr{S}, w)$. It follows from the continuity of $\varphi$ in its first argument, the fact that $\varphi$ in increasing in its first argument and the intermediate value theorem that there exists an outcome $y < w$ such that $\varphi(w, \mathscr{S}, w) = 0 = \varphi(y, A_k, w) + \varphi(y, \mathscr{S} \backslash A_k, w) + \varepsilon$. By P6, it follows that there exists a finite partition $\mathscr{P} = \{A_k : k = 1, ..., K\}$ such that $w \succ x_{A_k} y$ for all $A_k$ in $\mathscr{P}$. That is, $0 > \varphi(x, A_k, w) + \varphi(y, \mathscr{S} \backslash A_k, w)$. Hence $\varphi(x, A_k, w) < \varepsilon$, for all $A_k$ in $\mathscr{P}$, as required. ∎

The following example gives a preference relation that satisfies weak decomposability and each of Savage's first six postulates except P2. Yet, we can show this agent is not probabilistically sophisticated. In fact, this preference relation can accommodate the Ellsberg paradox.

EXAMPLE E. Let the state space $\mathscr{S}$ be the interval $[0, 1]$ with Lebesgue measure, and let $\{R, W, B\}$ be a partition of $S$ with $R = [0, 1/3)$ and $B = [2/3, 1]$. The set of outcomes $\mathscr{X}$ is taken to be $[0, 1]$. The preference relation $\succcurlyeq$ is represented by a function $V : \mathscr{L}_0 \to [0, 1]$, where $V$ is implicitly defined as $\int_0^1 \varphi(f(s), s, V(f)) \, ds = 0$, with

$$\varphi(x, s, v) = \begin{cases} x - v & \text{if} \quad (x > v \text{ and } s \in R) \quad \text{or} \quad (x \leqslant v \text{ and } s \notin R) \\ (1-a)(x-v) & \text{if} \quad (x \leqslant v \text{ and } s \in R) \quad \text{or} \quad (x > v \text{ and } s \notin R) \end{cases}$$

where $a \in (0, 1)$.

Let $\mu(A)$ denote the Lebesgue measure of any (measurable) event $A$. To see that the preference relation $\succcurlyeq$ is well-defined notice first that for any

act $f$ where neither $\mu(f^{-1}(1))$ nor $\mu(f^{-1}(0))$ is equal to one, we have $\int_0^1 \varphi(f(s), s, 0) \, ds = \int_0^{1/3} f(s) \, ds + \int_{1/3}^1 (1-a) f(s) \, ds > 0$, $\int_0^1 \varphi(f(s), s, 1) \, ds = \int_0^{1/3} (1-a) f(s) \, ds + \int_{1/3}^1 f(s) \, ds - 1 < 0$, and $\int_0^1 \varphi_v(f(s), s, v) \, ds < 0$. Hence for every act $f$ there exists a unique $v$ satisfying $\int_0^1 \varphi(f(s), s, v) \, ds = 0$.[22]

These preferences can be thought of as a state-dependent version of Gul's [16] disappointment aversion preferences in which, on the event $R$, "good" outcomes are relatively overweighted but, off $R$, "bad" outcomes are relatively overweighted. Since this preference relation conforms with the form in Proposition 6, it satisfies weak decomposability, P3 and P6.

To see that it also satisfies P4 observe that for any pair of outcomes $x > y$ and measurable event $A$, it follows from the implicit definition of $V(f)$ that

$$\mu(A \cap R)(x - V(x_A y)) + \mu(A \cap (\mathscr{S} \backslash R))(1-a)(x - V(x_A y))$$
$$+ \mu((\mathscr{S} \backslash A) \cap R)(1-a)(y - V(x_A y))$$
$$+ \mu((\mathscr{S} \backslash A) \cap (\mathscr{S} \backslash R))(y - V(x_A y)) = 0$$

Noting that $V(y) = y$, relatively straightforward algebraic manipulation yields

$$V(x_A y) - V(y) = \frac{[\mu(A \cap R) + \mu(A \cap (\mathscr{S} \backslash R))(1-a)]}{[1 - a(\mu(A \cap (\mathscr{S} \backslash R)) + \mu((\mathscr{S} \backslash A) \cap R))]} (x - y). \quad (4)$$

Given (4), for any pair of events $A$ and $B$, $V(x_A y) > V(x_B y)$ (that is, the individual prefers "betting on $A$" to "betting on $B$") if and only if

$$\frac{[\mu(A \cap R) + \mu(A \cap (\mathscr{S} \backslash R))(1-a)]}{[1 - a(\mu(A \cap (\mathscr{S} \backslash R)) + \mu((\mathscr{S} \backslash A) \cap R))]}$$
$$> \frac{[\mu(B \cap R) + \mu(B \cap (\mathscr{S} \backslash R))(1-a)]}{[1 - a(\mu(B \cap (\mathscr{S} \backslash R)) + \mu((\mathscr{S} \backslash B) \cap R))]}.$$

Since this inequality does not depend on the values of $x$ or $y$, Savage's P4 follows.

To see that this preference relation both violates P2 and is not probabilistically sophisticated, recall the choices in Ellsberg's [9] proposed experiment. An agent must bet on the draw of a ball from an urn containing red, white and blue balls. She only knows that a third of the balls are red. Let $R$ (respectively, $W$, $B$) be the event that the color of the drawn ball

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[22] For any act $f$ for which $\mu(f^{-1}(1))$ (respectively, $\mu(f^{-1}(0))$) is equal to one, $V(f)$ is naturally set to 1 (respectively, 0).

is red (resp. white, blue). Fixing two outcomes $x$ and $y$, $x > y$, the acts considered are

| Act | R | W | B |
|-----|---|---|---|
| $f' := x_R y$ | $x$ | $y$ | $y$ |
| $g' := x_W y$ | $y$ | $x$ | $y$ |

| Act | R | W | B |
|-----|---|---|---|
| $f'' := x_{R \cup B} y$ | $x$ | $y$ | $x$ |
| $g'' := x_{W \cup B} y$ | $y$ | $x$ | $x$ |

Ellsberg predicted that the typical agent would prefer $f'$ to $g'$, but prefer $g''$ to $f''$, thus exhibiting "uncertainty aversion." Such preferences violate P2 since $f'_{R \cup W} y \succ g'_{R \cup W} y$ but $g'_{R \cup W} x \succ f'_{R \cup W} x$. They also violate probabilistic sophistication since $f' \succ g'$ implies $\Pr(R) > \Pr(W)$, while $g'' \succ f''$ implies $\Pr(R) + \Pr(B) < \Pr(W) + \Pr(B)$.

The preferences given in Example E, however, are consistent with Ellsberg's prediction. To see this, let $v' := V(x_R y) = (x + 2y)/3$ and $v'' := V(x_{W \cup B} y) = (2x + y)/3$. We have, $\int_0^1 \varphi(g'(s), s, v') \, ds - \int_0^1 \varphi(f'(s), s, v') \, ds = -a(x - y)/9 < 0$, and $\int_0^1 \varphi(g''(s), s, v'') \, ds - \int_0^1 \varphi(f''(s), s, v'') \, ds = a(x - y)/9$. Hence, $V(x_R y) > V(x_W y)$ and $V(x_{R \cup B} y) < V(x_{W \cup B} y)$ as required.

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