ELSEVIER

Journal of Economic Behavior & Organization
Vol. 37 (1998) 277–290

JOURNAL OF
Economic Behavior
& Organization

# AUSI expected utility: An anticipated utility theory of relative disappointment aversion

Simon Grant*, Atsushi Kajii

*Australian National University Economics Program, RSSS ACT 0200, Australia*

Received 11 March 1996; received in revised form 9 September 1997; accepted 24 September 1997

**Abstract**

In this paper we provide an axiomatization for a representation of preferences over lotteries that is only one parameter richer than expected utility. Our model is a special case of Rank-Dependent Expected Utility. Moreover, we show that the same restriction on this parameter is required for: risk aversion; intuitive comparative static results for a reasonably general class of economically interesting choice problems; and accommodating some of the most well-known violations of Expected Utility Theory. © 1998 Elsevier Science B.V. All rights reserved.

*JEL classification*: D81

*Keywords*: Rank-dependent expected utility; Disappointment aversion; Allais paradox

## 1. Introduction

In recent years there has been a growing theoretical, empirical and experimental challenge to expected utility theory. Of the latter, the most famous and widely replicated are the original Allais (or common consequence) paradox and common ratio paradox. In spite of this, expected utility theory has remained the overwhelmingly dominant framework for modeling decision making under risk in Economics. Unsurprisingly, when faced with a veritable 'smorgasbord' of alternative non-expected utility theories, a researcher may quite naturally plump for the procedure that is well-understood and still well-accepted by the profession at large. Such a decision is made all the easier by researchers continually reporting new experimental findings that are inconsistent with all the major proposed alternatives. In addition, the researcher who ventures into the literature on non-expected utility, and decides upon a particular one, will generally find

* Corresponding author.


PII: S0167-2681(98)00094-8

little guidance as to what is an appropriate and parsimonious parameterization of the model.

A proclivity to experiment with an alternative to expected utility might well be encouraged if, first, that alternative is a generalization of Expected Utility Theory, thus, illustrating how the standard procedure may be incorporated as a special case of this more general approach. Secondly, the alternative should be able to accommodate at least some of the more well-known Allais-type paradoxes[1], and finally, the representation derived from 'intuitive' axioms should be as parsimonious an extension of expected utility as possible.

Thus, our motivation in this paper self-consciously mirrors that of Gul (1991) in his paper entitled '*A Theory of Disappointment Aversion*.' In that paper, Gul (1991) characterized preferences that could be specified by a class of functionals one parameter richer than expected utility. His model was a special case of preferences satisfying the betweenness property that satisfied the above desiderata.

Another versatile and useful generalization of expected utility is Rank-Dependent Expected Utility first introduced and axiomatized by Quiggin (1982), further expounded upon and extended by Allais (1988); Green and Jullien (1988); Segal (1987, 1989); Puppe (1990, 1991); Wakker (1993) and Yaari (1987). Our contribution in this paper is to provide an axiomatization of a family of rank-dependent expected utility preferences that satisfies Gul's three desiderata.

Formally, for any lottery $X$, whose support is a subset of $[0, M]$ and with an associated-cumulative distribution function $F_X$, the *rank-dependent* expected utility $V(X)$ can be expressed as:

$$V(X) = \int_{m=0}^{M} u(m)[f \circ F_X(\mathrm{d}m)], \tag{1}$$

where $f$ is an increasing function with $f(0) = 0$, and $f(1) = 1$. The family we characterize – we shall call it AUSI- (pronounced 'ozzie') Expected Utility – is the one for which the *probability transformation function* $f$ is a concave power function, that is, $f(p) = p^\alpha$ for some $\alpha$ in (0,1]. Hence it is an expected utility functional except for one extra distortion parameter $\alpha$. Moreover, the restriction on the values that $\alpha$ may take, entails that members of this family of preferences are risk averse as well as providing for intuitive comparative static results for a reasonably general class of economically interesting choice problems. This particular form of a rank-dependent expected utility functional has been used by Segal (1987) and was estimated from experimental data by Carbone and Hey (1994) and Hey and Orme (1995). But as far as we know, a proper characterization of the family is not known.

To motivate our axiom, consider the 'intuitive' axiom that Gul (1991) proposed to accommodate Allais paradox-type behavior which is based on a notion of 'disappointment aversion.' In Gul's paper, 'disappointment' in a lottery is a self-referenced concept. If the individual is indifferent between receiving outcome $x$ for sure or the lottery $X$, then

[1] Although as some researchers seem to suggest from their most recent findings, a simple and coherent model that accommodates all anomalies is appearing more and more like an unattainable 'Holy Grail' for choice theory under risk. See, for example, Camerer (1989); Loomes (1991) and Starmer (1992).

a 'disappointing' (respectively, 'elation') outcome for the lottery $X$ is one which is worse (respectively, better) than that certainty equivalent outcome $x$. Our related explanation of Allais paradox-type behavior is based on a notion we dub '*relative disappointment aversion*.' To motivate relative disappointment aversion and contrast it with Gul's approach, consider first the following pair of choice problems.

### 1.1. Common ratio effect

*PROBLEM 1*: Choose either $X_1$ or $Y_1$, where $X_1$ is a (degenerate) lottery that pays 3000 dollars for sure, and $Y_1$ is a lottery that pays 4000 dollars with probability 0.8 and zero dollars with probability 0.2.

*PROBLEM 2*: Choose either $X_2$ or $Y_2$, where $X_2$ is a lottery that pays 3000 dollars with probability 0.25 and zero dollars with probability 0.75, and $Y_2$ is a lottery that pays 4000 dollars with probability 0.2 and zero dollars with probability 0.8.

Many studies have shown a systematic tendency for subjects when faced with such problems to express a preference for $X_1$ over $Y_1$ in the first problem and $Y_2$ over $X_2$ in the second, constituting a violation of the implication of expected utility theory. In particular, the Independence axiom states that for any three pair-wise independent lotteries $X$, $Y$ and $Z$, and fraction $\gamma$ in (0,1), $X$ is preferred to $Y$ implies $\gamma X + (1-\gamma)Z$ (the $\gamma, 1-\gamma$ probability mixture of $X$ and $Z$) is preferred to $\gamma Y + (1-\gamma)Z$ (the corresponding $\gamma, 1-\gamma$ probability mixture of $Y$ and $Z$). Notice if we let $Z^-$ denote the degenerate lottery, zero dollars for sure, we can express $X_2$ as $\frac{1}{4}X_1 + \frac{3}{4}Z^-$ and $Y_2$ as $\frac{1}{4}Y_1 + \frac{3}{4}Z^-$. Thus, the above pattern of choice directly violates Independence. Gul offers the explanation that $X_1$ offers the individual a lower probability of 'disappointment' than $Y_1$ (zero vs. 0.2). Mixing both the lotteries with $Z^-$, increases the probability of 'disappointment' proportionately more for the mixture $\frac{1}{4}X_1 + \frac{3}{4}Z^-$ than for the mixture $\frac{1}{4}Y_1 + \frac{3}{4}Z^-$, hence if $X_1$ was only 'marginally' preferred to $Y_1$, both Gul's disappointment aversion and our relative disappointment aversion notions suggest that the preference may be reversed after mixing with an inferior lottery. Moreover, if the agent was indifferent between $X_1$ and $Y_1$ then strict versions of both Gul's and our notions would imply $\frac{1}{4}Y_1 + \frac{3}{4}Z^-$ is strictly preferred to $\frac{1}{4}X_1 + \frac{3}{4}Z^-$.

Our point of departure from Gul's analysis is to ask: would a similar preference reversal occur if lotteries were mixed with a superior outcome? In Gul's analysis the answer might well be yes. To see why, consider the following problem.

*PROBLEM 3*: Choose either $X_3$ or $Y_3$, where $X_3$ is a lottery that pays 3000 dollars with probability 0.25 and 4000 dollars with probability 0.75, and $Y_3$ is a lottery that pays 4000 dollars with probability 0.95 and zero dollars with probability 0.05.

Letting $Z^+$ denote the degenerate lottery, 4000 dollars for sure, notice that mixing both lotteries with $Z^+$ reduces the probability of 'disappointment' for the mixture $\frac{1}{4}Y_1 + \frac{3}{4}Z^+ [= Y_3]$ but increases it for the mixture $\frac{1}{4}X_1 + \frac{3}{4}Z^+ [= X_3]$. If $X_1$ was only 'marginally' preferred to $Y_1$, then Gul's disappointment aversion suggests that the preference may again be reversed, now after mixing with a superior lottery.[2] Moreover, if

[2] More generally, mixing both lotteries with a superior outcome increases the probability of 'disappointment' more for the mixture with $X_1$ than for the mixture with $Y_1$.

the agent was indifferent between $X_1$ and $Y_1$ then strict disappointment aversion implies $\frac{1}{4}Y_1 + \frac{3}{4}Z^+$ is strictly preferred to $\frac{1}{4}Y_1 + \frac{3}{4}Z^+$.[3]

Our conjecture, on the other hand, is that an individual who expressed a preference for $X_1$ over $Y_1$ (respectively, $Y_1$ over $X_1$) would prefer from $X_3$ to $Y_3$ (respectively, from $Y_3$ to $X_3$), irrespective of his or her preference between $Y_2$ and $X_2$. Loosely speaking, this is what we shall refer to as the Axiom of Upward Scale Invariance (AUSI). Our reasoning is that 'disappointment' occurs whenever a lottery does not pay its best outcome, not just when the outcome realized is worse than the certainty equivalent outcome for that lottery. Hence mixing with a superior outcome does not change the relative likelihood between $X_1$ and $Y_1$ of receiving any outcome worse than the best outcome either $X_1$ or $Y_1$ offers. In this sense, we argue, the relative likelihood of disappointment between $X_1$ and $Y_1$ and pairs of lotteries obtained by mixing $X_1$ and $Y_1$ with superior outcomes is unchanged, and hence the preference will not be reversed.

In Section 2, we define notation, formally state AUSI and derive the AUSI-expected utility functional representation of preferences. The formal proof appears in the appendix. Section 3 demonstrates how the same restriction on this parameter is required for; risk aversion, intuitive comparative static behavior, as well as accommodating the above common ratio effect. We conclude in Section 4.

## 2. The Axiom of Upward Scale Invariance and the AUSI-EU representation

We assume that the outcomes all lie in a bounded interval of real numbers, [0, $M$]. Let the space of measures (i.e. lotteries) over this outcome set be denoted $\mathscr{L}([0,M])$. For every $X$ in $\mathscr{L}([0,M])$, define: the cumulative distribution function $F_X$ by $F_X(m) = \Pr(X \leq m)$ and let supp($X$) denote the support (range) of $X$. It will prove convenient to work also with the decumulative distribution function $G_X \equiv 1 - F_X$. A lottery that places all probability weight on a single outcome, say $x$ in [0, $M$], will be denoted by $\delta_x$.

Preferences on $\mathscr{L}([0,M])$ are described by a binary relation $\succeq$. As is usual, $\succ$ and $\sim$ denote the symmetric and asymmetric parts of the relation, respectively. We assume that $\succeq$ respect the first order stochastic dominance, and that $\succeq$ can be represented by what Chew and Epstein (1989) call a *Rank-Linear Utility* functional.

**Definition**. $\succeq$ admits an rank-linear utility representation if there exists an atomless, positive finite measure φ on $[0,M] \times [0,1]$ with φ (A)>0 for every non-empty open set $A \subset [0,M] \times [0,1]$, such that for all $X$ and $Y$ in $\mathscr{L}([0,M])$: $X \succeq Y$ if and only if

[3] This can be seen if the three pairs of lotteries ($X_i$,$Y_i$), $i$=1–3, are plotted in a Marshak–Machina triangle where the $x$-coordinate (respectively, $y$-coordinate) corresponds to the probability the lottery assigns to the outcome zero (respectively, 4000 dollars) with the remaining probability assigned to the outcome 3000 dollars. As Gul's style preferences satisfy 'betweenness,' the indifference curves in this triangle are all linear. Strict disappointment aversion implies the indifference curves in the region of lotteries that are worse than (respectively, better than) $X_1$ 'fan-out' (respectively, 'fan-in'). Hence, if $X_1$ and $Y_1$ lie on the same indifference curve, then from the geometry of the indifference map it follows that $Y_2$ (respectively, $Y_3$) is strictly preferred to $X_2$ (respectively, $X_3$).

$$\int_{m=0}^{M}\int_{p=0}^{G_X(m)}\varphi(\mathrm{d}m,\mathrm{d}p)\geq\int_{m=0}^{M}\int_{p=0}^{G_Y(m)}\varphi(\mathrm{d}m,\mathrm{d}p). \quad (2)$$

As Chew and Epstein (1989) note, rank-linear utility functionals are distinguished by the rank ordering of outcomes prior to the application of the representation. Axiomatizations of rank-linear utility repesentations can be found in Chew and Epstein (1989) (but see also Chew et al., 1993), Green and Jullien (1988) or Segal (1989, 1993). Note that if $\succeq$ admits a rank-linear utility representation, then $\succeq$ is a continuous relation with respect to the weak topology, and respects first order stochastic dominance. However, $\varphi$ need not to be absolutely continuous with the product Lebesgue measure (see Wakker, 1993).

Rank-linear utility preferences and preferences that exhibit the betweenness property (i.e. probability mixtures of two lotteries lie in preference between those two lotteries) are the two most widely studied and applied axiomatized generalizations of expected utility. In rank-linear utility theory, the admission of such a measure representation may be viewed as treating (ranked) outcomes and probabilities in an analogous fashion. An expected utility maximizer corresponds to the special case of a product measure where the marginal measure over probabilities is Lebesgue. More general measure representations have been used to resolve the original Allais (or common consequence) paradox as well as the common ratio paradox discussed in the introduction (see Quiggin, 1982; Segal, 1987; Allais, 1988). Possible explanations for the 'preference reversal' phenomenon (Karni and Safra, 1987), the Friedman–Savage paradox of the joint purchase of insurance and actuarial unfair lottery tickets (Green and Jullien, 1988; Puppe, 1991; Quiggin, 1991), and even the Ellsberg paradox (Segal, 1987) have also been provided by different variants of this model.

With the exception of Green and Jullien and Puppe, all the above applications have employed a product measure in their representations. If $\varphi$ is a product measure then integrating by parts each side of Eq. (2) yields an rank-dependent expected utility representation as expressed in Eq. (1) for the preference relation. As Segal (1990) admits, the axiom that he provided in his 1987 paper to ensure the measure was a product measure lacks normative appeal.[4] By axiomatizing two-stage lotteries without the reduction of compound lottery axiom, Segal (1990), however, obtains a product measure incidentally as the consequence of the two-stage lottery preferences exhibiting a first-order stochastic dominance principle. Both Quiggin (1982) and Yaari (1987) derive produce measures for one-stage lottery preferences from more behaviorally compelling axioms but Quiggin's additional requirement of $f(0{\cdot}5)=0{\cdot}5$ rules out risk averse preferences except for the special case of expected utility and Yaari's utility function is linear, that is, $u(x)=x$. The most general derivation to date of an rank-dependent expected utility representation is supplied by Wakker (1994) but his approach of working with 'derived tradeoffs' is distinct from the rest of the literature. Thus, we think it is more natural to start from the general rank-linear utility representation.

Our approach is to maintain the homothetic property (with respect to superior outcomes) of the indifference map in the probability simplex implied by the

[4] Chateauneuf (1990) also provides an axiom on preferences that implies the multiplicative separability of the measure, but again it is difficult to discern the behavioral implications of his axiom.

independence axiom, but relax the linearity property. We believe the nice technical contribution of this paper is that by maintaining this homotheticity property when the outcome set is a continuum, we show that the measure for rank-linear utility representation must be a product measure and thus a member of the class of rank-dependent expected utility functionals, and moreover, that it is only one parameter richer than expected utility.

**Definition**. AUSI (Axiom of Upward Scale Invariance): For all $X$ and $Y$ in $\mathscr{L}([0, M])$ and all $z$ in $[0, M]$ such that $z \succeq x$ for all $x$ in supp($X$) $\cup$ supp($Y$):$X \succeq Y \Rightarrow \gamma X + (1-\gamma)\delta_z \succeq \gamma Y + (1-\gamma)\delta_z$ for all $\gamma$ in $(0, 1)$.

**Proposition 1**. *Suppose admits an rank-linear utility representation. Then $\succeq$ satisfies AUSI if and only if $\succeq$ can be represented by an AUSI-expected utility functional $V(X) = \int u(m)d[F_X(m)^\alpha]$, where $\alpha$ >0: that is,*

$$X \succeq Y \Leftrightarrow \int_{m=0}^{M} u(m)\mathrm{d}[F_X(m)^\alpha] \geq \int_{m=0}^{M} u(m)\mathrm{d}[F_Y(m)^\alpha]. \tag{3}$$

It is immediately apparent that expected utility corresponds to the case where $\alpha$ equals one.

By integrating by parts, and from the relation $F_X(m)^\alpha = (1 - G_X(m))^\alpha = \int_0^{G_X(m)} d(1-(1-p)^\alpha)$, we obtain the following alternative form of the representation for preferences specified in Eq. (3).

$$V(X) = \int_{m=0}^{M} [1 - [1 - G_x(m)^\alpha]\mathrm{d}[u(m)], \tag{4}$$

$$= \int_{m=0}^{M} \left( \int_{p=0}^{G_X(m)} \mathrm{d}[1-(1-p)^\alpha] \right) \mathrm{d}[u(m)], \tag{5}$$

$$= \int_{m=0}^{M} \left( \int_{p=0}^{G_X(m)} \alpha(1-p)^{\alpha-1}\mathrm{d}p \right) \mathrm{d}[u(m)]. \tag{6}$$

Thus, Proposition 1 provides a characterization of rank-dependent expected utility preferences that are also homothetic 'in probabilities' with respect to superior outcomes. Eq. (3) is a representation characterized by the usual utility function over outcomes, $u(m)$, and a continuous and strictly increasing *probability transformation* of the cumulative distribution function, $f(p) = p^\alpha$, while Eq. (4) illustrates an alternative formulation that is characterized by the same utility function over outcomes and a continuous and strictly increasing probability transformation of the decumulative distribution function, $g(p) = 1 - (1-p)^\alpha$.[5,6]

[5] Notice that $g(p) \equiv 1 - f(1-p)$ which is consistent with the definition $G_X(m) \equiv 1 - F_X(m)$.

[6] Our representation should be distinguished from Puppe's (1991) specificiation of *homogeneous rank-dependent utility*. In his context, homogeneity in the probabilities refers to the utility functional over the set of *elementary* lotteries. These are lotteries whose support have at most two elements one of which must be the 'status quo' outcome usually referred to as 0. Hence without further assumptions, this homogeneity requirement does not imply any homothetic properties for the indifference map in the probability simplex.

To see the sufficiency of Proposition 1, fix any lottery $X$ and any outcome $z$ such that $z \geq m$ for all $m$ in supp $X$. Notice that for any $\lambda$ in (0,1) and any $m$ in supp $X, F_{\lambda X+(1-\lambda)\delta_z}(m) = \lambda F_X(m)$ (and hence, $1 - G_{\lambda X+(1-\lambda)\delta_z}(m) = \lambda[1 - G_X(m)]$). Hence from Eq. (4) it follows that: $V(\lambda X + [1-\lambda]\delta_z) = \lambda^\alpha V(X) + (1-\lambda^\alpha)u(z)$. So, $V(X) \geq V(Y)$ if and only if $V(\lambda X + [1-\lambda]\delta_z) \geq V(\lambda Y + [1-\lambda]\delta_z)$. The necessity part is more complicated and a proof can be found in the appendix.

The natural counterpart to AUSI is homotheticity with respect to *inferior* outcomes, that is, the preference between a pair of lotteries is maintained if we transform that pair by mixing them with an outcome that is (weakly) inferior to any outcome in the support of either of those two lotteries.

**Definition**. ADSI (Axiom of Downward Scale Invariance): For all $X$ and $Y$ in $\mathscr{L}([0, M])$ and all $z$ in [0, $M$] such that $x \succeq z$ for all $x$ in supp($X$)∪ supp($Y$):

$$X \succeq Y \text{ implies } \gamma X + (1-\gamma)\delta_z \succeq \gamma Y + (1-\gamma)\delta_z \text{ for all } \gamma \text{ in } (0,1).$$

The analog to Proposition 1 is that a preference relation that admits an rank-linear utility representation and satisfies ADSI can be represented by an rank-dependent expected utility function with a probability transformation of the cumulative distribution function of the form, $f(p) = 1-(1-p)^\beta$, $\beta > 0$, or equivalently a probability transformation of the decumulative distribution function of the form, $g(p) = p^\beta$. Clearly, if a preference relation satisfies both AUSI and ADSI, then $\alpha = \beta = 1$, that is, the preference relation represents an expected utility maximizer. More significantly for the motivation and focus of this paper is that ADSI is explicitly ruled out by the choice patterns of the common ratio paradox discussed in Section 1.

## 3. Risk aversion, comparative statics and paradox resolutions

Chew et al. (1987) show that one rank-dependent expected utility maximizer is more risk averse (in the Diamond and Stiglitz, 1974 sense) than another rank-dependent expected utility maximizer if and only if the former's utility index is a concave transformation of the latter's and the former's probability transformation function is a concave transformation of the latter's. Hence rank-dependent expected utility maximizer is risk averse (i.e. more risk averse than an expected value maximizer) if and only if both his or her utility index and probability transformation function are concave. Hence for two AUSI-expected utility functionals $V^*$ and $V$, with respective utility indexes $u^*$ and $u$, and probability transformation parameters $\alpha^*$ and $\alpha$, $V^*$ are more risk averse than $V$ if and only if $u^*$ is a concave transformation of $u$ and $\alpha^* \leq \alpha$.[7] And risk aversion thus requires that the probability transformation parameter to be less than or equal to 1.

[7] Notice that we cannot directly apply Theorem 1 of Chew et al. (1987 p.374) since they require that the RDEU functionals are Gateux differentiable (see Section 2, pp. 372–373, for definition and explanation). For AUSI-EU, if $\alpha$ <1 then the AUSI-EU functional is not Gateux differentiable since $f'(0) \equiv \frac{d}{dp}[p^\alpha]\,|_{p=0}$ does not exist (see Chew et al., 1987; Grant and Kajii (1994), Lemma 1 and Corollary 1, p.373). The implication of their theorem still holds, however, for AUSI-EU functionals. The interested reader is referred to our earlier discussion paper where the extension of Chew et al., 1987 to AUSI-EU is formally proved.

Chew et al. (1987) noted that since the transformation of the cumulative distribution function is defined independently of the outcomes for the rank-dependent expected utility representation, the definitions of decreasing (increasing and constant) absolute and relative risk aversion involve only properties of the utility index, $u$. Therefore, if the utility index $u$ satisfies those properties in the sense of Arrow-Pratt, then the AUSI-expected utility functional $V$ is also said to exhibit those properties.

The economic relevance of notions such as risk aversion and comparisons made therein, ultimately stems from the insights they can provide for the effects that changing different parameters have on the choice behavior of a decision maker in an environment involving risk. Chew et al. (1987) illustrated the usefulness of the above concepts of risk aversion in a simple portfolio choice of allocating wealth between two assets, one with a guaranteed return and the other with a random return. Quiggin (1995) extended this analysis to Feder (1977) quite general specification of a comparative static problem for choice under uncertainty.

To adapt Quiggin's framework to ours, first, let $\tilde{X}_0$ and $\tilde{W}_0$ be two random variables both defined as non-decreasing functions from a state space [0,1] with Lebesgue measure into a set of outcomes $[-m_1, m_1]$. Assume both have an expected value of zero and a standard deviation of one. The decision maker's possibly random base wealth is determined by the random variable $\tilde{W} = \mu_\omega + \sigma_x \tilde{W}_0$, where $\mu_\omega$ and $\sigma_x$ are non-negative constants. The return of the risky activity which the decision maker can undertake is described by the random variable $\tilde{X} = \mu_x + \sigma_x \tilde{X}_0$, where $\mu_x$ and $\sigma_x$ are also non-negative constants. The cost of undertaking a given level of the risky activity is given by an increasing cost function $C(\cdot)$ with $C(0) = 0$. Thus, the random variable $a\tilde{X} - C(a) + \tilde{W}$ describes the return of final wealth enjoyed by the decision maker if he or she chooses a level $a$ of the risky activity. Notice that in this set-up the state space [0,1] is naturally ordered in the sense that $s > s'$ means that state $s$ leads to an outcome no worse than state $s'$ for both $\tilde{W}_0$ and $\tilde{X}_0$ and hence for $\tilde{X}, \tilde{W}$ and $a\tilde{X} - C(a) + \tilde{W}$ as well. If for any random variable $\tilde{Z}$, we let $[Z]^L$ denote the lottery over outcomes induced by $\tilde{Z}$, then the general problem can be formulated as:

$$\max_{a \in A} V([a\tilde{X} - C(a) + \tilde{W}]^L), \tag{7}$$

where $\tilde{X} = \mu_x + \sigma_x \tilde{X}_0,\ \mu_x > 0, \sigma_x > 0$; $\tilde{W} = \mu_\omega + \sigma_\omega \tilde{W}_0,\ \mu_\omega > 0, \sigma_\omega > 0$; and $A \subset \{a \geq 0 : [a\tilde{X} - C(a) + \tilde{W}]^L \in \mathscr{L}([0, M])\}$.

The simple portfolio choice problem corresponds to $\sigma_\omega = 0$, $C(a) = a$, $A = [0, \mu_\omega]$ and $[\tilde{X}]^L$ in $\mathscr{L}([0, M/\mu_\omega])$, for some $\mu_\omega$ in [0, $M$]. Thus, the action $a$ corresponds to the decision of how much of the initial wealth is to be placed in the risky asset $X$, while the cost represents the foregone opportunity cost of not investing in the riskless asset with a normalized return of 1. Alternatively, if we interpret: $\tilde{X}$ as the random price of a product; $a$ as the quantity produced of this good at cost $C(a)$ by a perfectly competitive AUSI-EU maximizing firm; and $\tilde{W}$ as the (possibly random) profits for this firm from its other business operations; the argmax of the problem specified in Eq. (7) corresponds to the supply schedule of this AUSI-EU maximizing firm.

For the problem specified in Eq. (7) we can consider the effects on the optimal choice of action of changes in the parameters $\mu_\omega, \sigma_x, \mu_x$ and $\sigma\omega$. Increases in $\mu_\omega$ and $\mu_x$

correspond to ‘income’ effects on base wealth and the risky activity, respectively. As individuals become wealthier it is common to assume that they become more willing to face risks and so we should expect the level of their risk-taking activity to increase. As Chew et al. (1987) (Theorem 2) show for rank-dependent expected utility maximizers (and hence for AUSI-expected utility and expected utility maximizers as well) decreasing absolute risk aversion is indeed the necessary and sufficient condition for the decision maker to be willing to engage in more risky activity levels as his or her wealth increase.

Increases in $\sigma_x$ and $\sigma_\omega$ represent a ‘multiplicative spread’ increase in the riskiness of risky activity and the base wealth, respectively. Given decreasing absolute risk aversion, Quiggin (1993) (Proposition 4) shows that such a multiplicative spread increase in risk inhibits the level of the risk activity undertaken by an rank-dependent expected utility maximizer if and only if the probability transformation function $f$ satisfies $f(p) \geq p$ for all $p$ in [0,1]. That does not coincide with the condition on $f$ for risk aversion for a general rank-dependent expected utility functional, but it does coincide for an AUSI-expected utility maximizer.

Collecting these results together in the next proposition demonstrates how the definitions of risk aversion described in the subsection above correspond to intuitive comparative static results for an AUSI-expected utility decision maker facing the problem in Eq. (7).

**Proposition 2**. *Let $a^*$ be the optimal value of a, for an AUSI-expected utility functional V in the control problem Eq. (7). Let u be the utility function and $\alpha$ be the probability transformation parameter for V. If u satisfies decreasing absolute risk aversion, in the sense of Arrow-Pratt, then*

*$a^*$ is non-decreasing for increases in $\mu_\omega$;*
*$a^*$ is non-decreasing for increases in $\mu_x$;*
*$a^*$ is non-increasing for increases in $\sigma_x$ if and only if $\alpha \leq 1$;*
*$a^*$ is non-increasing for increases in $\sigma_\omega$ if and only if $\alpha \leq 1$.*

Finally, let us first return to the two pairs of lotteries that constituted the Common Ratio Effect. Recall that the four lotteries from the introduction were: $X_1$=(3,000,1); $Y_1$=(0,0.2; 4,000,0.8); $X_2$=(0,0.75; 3,000,0.25) and $Y_2$=(0,0.8; 4,000,0.2). The common ratio effect corresponded to the preference pattern $X_1 \succ Y_1$ and $Y_2 \succ X_2$. Say $\succeq$ can be represented by an AUSI-expected utility functional $V$, with $u(0)=0$ and $u(4{,}000)=1$ and probability transformation parameter $\alpha$. Applying Eq. (3) we have: $V(X_1) > V(Y_1)$ if and only if $u(3,000) > [1-(0.2)^\alpha]$, and $V(Y_2) > V(X_2)$ if and only if $[1-(0.8)\alpha] > [1-(0.75)^\alpha]u(3,000)$. Or $[1-(0.8)^\alpha]/[1-(0.75)^\alpha] > u(3,000) > [1-(0.2)^\alpha]$. So, the paradox is resolved if and only if $[1-(0.8)^\alpha] > 1-(0.75)^\alpha-(0.2)^\alpha+(0.15)^\alpha$, which holds if and only if $\alpha < 1$. Notice that the slightest deviation from expected utility ($\alpha = 1$) can explain the common ratio effect.[8]

[8] Segal (1987) also shows that a generalization of the original Allais (common consequence) paradox can be accommodated by an RDEU functional if and only if the probability transformation function is concave which of course again corresponds to $\alpha \leq 1$ for an AUSI-EU functional.

## 4. Concluding remarks

To conclude, let us suggest what might provide fruitful lines for further research on AUSI-expected utility. First, experiments could be conducted that involve subjects facing decision problems 1 through 3 described in Section 1 or variants thereof to examine how robust a theory AUSI-expected utility is. Our own (very) informal tests suggest that most decision makers make choices that are consistent with AUSI-expected utility for problems Eqs. (1) and (3). Although one should note that some of the evidence from the experimental studies of Camerer (1989) and Starmer (1992), and much of that from Loomes (1991) is inconsistent not only with expected utility but with risk averse rank-dependent expected utility and hence risk averse AUSI-expected utility as well. The evidence presented by Wu (1994) goes further in reporting choice patterns that are inconsistent with any member of the larger class of rank-linear utility preferences.

For situations in which rank-dependent expected utility is deemed an appropriate model for choice under risk, the form of the AUSI-expected utility functional does seem to open the way for parametric estimation of rank-dependent expected utility functionals. For general rank-dependent expected utility functionals, even when $u$ is given, the parameter space is still the set of all transformation functions, that is, an infinite dimensional parameter space. To apply statistical estimation techniques, it is desirable to confine oneself to a finite dimensional parameter space. The set of AUSI-expected utility, which can be seen as a one-dimensional subspace of the infinite dimensional space of distortion functions, seems to be a natural candidate for estimation. See, for example, Carbone and Hey (1994) and Hey and Orme (1995) where AUSI-expected utility functionals along with other tightly parameterized functional forms were estimated using experimentally generated data.

Another attractive feature of an AUSI-expected utility functional (which is a feature common to all rank-dependent expected utility functionals) is that the preference relation it represents is consistent with Rubinstein's 'ychologically plausible' procedure for preference determination based on similarity relations defined on probabilities and outcomes (see Rubinstein, 1988, Proposition 3, p.152).[9] On the other hand, as our outcomes represent the decision makers 'final' or 'absolute' position, our model is not formally consistent with models popular in the psychology literature for which changes in the outcome level, measured relative to a given 'status quo,' are the 'carriers of value.' Even for these models such as Luce and Fishburn (1991) rank- and sign-dependent model or Tversky and Kahneman (1992) cumulative prospect theory, AUSI-expected utility may be applied as a parsimoniously parameterized family of functionals for situations where lotteries with only non-negative changes are possible. A possible extension to lotteries that contain positive and non-negative changes, is to require AUSI only for pairs of lotteries with non-negative changes, and ADSI for pairs of lotteries with non-positive

[9] Rubinstein only considers elementary lotteries. These are lotteries whose support contain at most two outcomes one of which is the 'worst' possible outcome denoted as 0. For this set of lotteries, if we normalize $u(x) = 0$, then the AUSI-expected utility evaluation of the elementary lottery $(x, p; 0, 1-p)$ is $(1 - [1-p]^{\alpha})u(x)$ which is multiplicatively separable in the non-zero outcome and the probability associated with that outcome.

changes.[10] This would lead to a cumulative prospect theory functional with weighting function $f^{-}(p) = 1-(1-p)^{\alpha}$ applied to losses and $f^{+}(p) = p^{\alpha}$ to gains. It should be noted that the experimental evidence is generally consistent with a weighting function for both $f^{-}$ and $f^{+}$, that is, $S$-shaped with a concave region for small $p$, and a convex region for large $p$ (see Tversky and Kahneman, 1992). We aim in future research to provide a characterization of a parsimoniously parameterized family of $S$-shaped weighting functions utilizing an axiom similar in nature and intuitive appeal to AUSI.

Although the previous paragraph has highlighted some of the weaknesses and limitations of the AUSI-expected utility representation, let us finish by restating that the beauty of AUSI-expected utility for applications is its tractability. At the theoretical level we have already demonstrated that comparative static results that can be derived from expected utility and can be naturally extended to AUSI-expected utility. We believe such natural extensions can be found in many areas where expected utility is currently employed.

## Appendix

**Proof of Proposition 1**. Consider the following pair of lotteries. For $1 > q_X > q_Y > r_Y > r_X > 0$ and $0 < z < M$, let

$$X = (\$0 \text{ with probability } 1-q_X,\ \$z \text{ with prob. } q_X - r_X,\ \$M \text{ with prob. } r_X)$$

$$Y = (\$0 \text{ with probability } 1-q_Y,\ \$z \text{ with prob. } q_Y - r_Y,\ \$M \text{ with prob. } r_Y)$$

Since preferences admit an rank-linear utility representation, there exists an atomless, positive finite measure φ on $[0,M]\times[0,1]$ such that: $X \sim Y$ if and only if

$$\int_{m=0}^{z}\int_{p=qY}^{q_X}\varphi(\mathrm{d}m,\mathrm{d}p) = \int_{m=z}^{M}\int_{p=r_X}^{r_Y}\varphi(\mathrm{d}m,\mathrm{d}p); \tag{8}$$

that is, if and only if the measure over the rectangle $[0,z]\times[q_Y,q_X]$ (i.e. where $F_X$ is less than $F_Y$) is equal to the measure over the rectangle $[z,M]\times[r_X,r_Y]$ (i.e. where $F_X$ is greater than $F_Y$). If $X\sim Y$ then AUSI implies that for all $\lambda$ in (0,1] we have:

$$\int_{m=0}^{z}\int_{p=1-\lambda+\lambda q_Y}^{1-\lambda+\lambda q_X}\varphi(\mathrm{d}m,\mathrm{d}p) = \int_{m=z}^{M}\int_{p=1-\lambda+\lambda r_X}^{1-\lambda+\lambda r_Y}\varphi(\mathrm{d}m,\mathrm{d}p). \tag{9}$$

We shall re-write Eqs. (8) and (9) by change of variables, so that the scale parameter $\lambda$ enters additively. Consider a continuous transformation $(m,p)\mapsto(m,-\ln(1-p))$ which maps $[0,M]\times[0,1]$ to $[0,M]\times[0,\infty]$ bijectively. Denote the induced measure by $\nu$ which is atomless, finite and positive, thus, without loss of generality, assume $\nu$ is a probability measure. By writing $s=-\ln(1-p)$, $a=-\ln(1-q_Y)$, $b=-\ln(1-q_X)$, $c=-\ln(1-r_X)$, $d=-\ln(1-r_Y)$, $t=-\ln\lambda$, Eqs. (8) and (9) become: $X\sim Y$ if and only if for all $t\geq 0$,

[10] This seems consistent with the findings of Kahneman and Tversky (1979) for 'common ratio' pairs of lottery choices involving losses rather than gains.

$$\int_{m=0}^{z}\int_{s=a+t}^{b+t}\nu(\mathrm{d}m,\mathrm{d}s)=\int_{m=z}^{M}\int_{s=c+t}^{d}+t\nu(\mathrm{d}m,\mathrm{d}s). \tag{A.1}$$

That is, after the change of variables, the induced measure has the following property: $X\sim Y$ now corresponds to the measure over the rectangle $[0,z]\times[a,b]$ being equal to the measure over the rectangle $[z,M]\times[c,d]$, and AUSI requires that translating these two rectangles up by the same amount $t$ keep them equal. We shall show that this 'translation invariance' property Eq. (A.1) leads to the AUSI-expected utility functional.

Fix $z$ in $(0, M)$ arbitrarily. For any $x\geq 0$, define $f(z,x)=\int_{m=0}^{z}\int_{s=x}^{\infty}\nu(\mathrm{d}m,\mathrm{d}s)$ and $g(z,x)=\int_{m=z}^{M}\int_{s=x}^{\infty}\nu(\mathrm{d}m,\mathrm{d}s)$, which are distribution functions of measure $\nu$. Since $\nu$ is atomless, $f$ and $g$ are well-defined, continuous, positive, decreasing in their second argument and differentiable almost everywhere (a.e.). Given $(\mathrm{d}p/\mathrm{d}s)$ $=\mathrm{e}^{-s}$, we are done if we show that $f$ has the form $\int_{m=0}^{z}\left(\int_{s=x}^{\infty}\alpha\mathrm{e}^{-sa}\mathrm{d}s\right)\mathrm{d}[u(m)]=u(z)\mathrm{e}^{-x\alpha}+\psi(z)$ with $\alpha>0$, $u(0)=0$ and $u(M)+\psi(M)=1$, since

$$\int_{m=0}^{M}\int_{p=0}^{G_X(m)}\varphi(\mathrm{d}m,\mathrm{d}p)=\int_{m=0}^{M}[f_z(\mathrm{d}m,0)-f_z(\mathrm{d}m,-\ln(1-G_X(m)))].$$

For ease of exposition we will suppress for the moment the first argument of $f$ and $g$. We can take $f'$ to be measurable function with $f'<0$ a.e. (see, e.g. Royden, 1988 for these facts).

Since $f$ and $g$ are positive, there exists $\overline{h}>0$ such that for each pair $h_1,\ h_2<\overline{h}$, there exists a unique number $\zeta(h_1,\ h_2)>0$ satisfying:

$$g(0)-g(h_1)=f(h_1+h_2)-f(h_1+h_2+\zeta(h_1,\ h_2)). \tag{A.2}$$

Or, equivalently, $\zeta(h_1,\ h_2)=-f^{-1}(g(0)-g(h_1)-f(h_1+h_2))-h_1-h_2$. Thus, $\zeta$ is continuous, and (partially) differentiable a.e. where it is defined. By Eq. (A.1) (i.e. AUSI) and the construction of $f$ and $g$, for any $x\geq 0$,

$$g(x)-g(x+h_1)=f(x+h_1+h_2)-f(x+h_1+h_2+\zeta(h_1,h_2)). \tag{A.3}$$

Choose any $x\geq 0$, and $h_1$, $h_2$ in $(0,\overline{h})$ such that $x$ and $x+h_1$ are points of differentiability of $g$, and $x+h_1+h_2$ is a point of differentiability of $f$, and $(h_1,\ h_2)$ is a point of differentiability of $\zeta$. By differentiating Eq. (A.3) with respect to $h_1$ and $h_2$, we have:

$$g'(x)-g(x+h_1)=f'(x+h_1+h_2)-f'(x+h_1+h_2+\zeta(h_1,h_2))\left(1+\frac{\partial}{\partial h_1}\zeta(h_1,h_2)\right). \tag{A.4}$$

$$f'(x+h_1+h_2)=f'(x+h_1+h_2+\zeta(h_1,h_2))\left(1+\frac{\partial}{\partial h_2}\zeta(h_1,h_2)\right). \tag{A.5}$$

Note that $(1+\partial\zeta/\partial h_2)\neq 0$ a.e. since $f'$ is positive a.e.. So from Eqs. (A.4) and (A.5), we have a.e.:

$$g'(x+h_1)=f'(x+h_1+h_2)[(1+\partial\zeta/\partial h_1)/(1+\partial\zeta/\partial h_2)-1]. \tag{A.6}$$

Let $L(h_1,h_2)=[(1+\partial\zeta/\partial h_1)/(1+\partial\zeta/\partial h_2)-1]$. Fix one $(h_1^*,h_2^*)$ small enough. Then from Eq. (A.6)$g'(x+h_1^*)=f'(x+h_1^*+h_2)L(h_1^*,h_2)$ and $g'(h_1^*)=f'(h_1^*+h_2)$

$L(h_1^*, h_2)$. So, $g'(x+h_1^*)f'(h_1^*+h_2) = f'(x+h_1^*+h_2)g'(h_1^*)$ or:

$$f'(x+h_1^*+h_2^*)L(h_1^*,h_2^*)f'(h_1^*+h_2) = f'(x+h_1^*+h_2)g'(h_1^*). \tag{A.7}$$

Set $\tilde{f}(x) \equiv f'(x+h_1^*+h_2^*)L(h_1^*,h_2^*)/g'(h_1^*)$, which is measurable in $x$. Then:

$$\begin{aligned}\tilde{f}(x)\tilde{f}(h_2-h_2^*) &= f'(x+h_1^*+h_2^*)f'(h_1^*+h_2)[L(h_1^*,h_2^*)/g'(h_1^*)]^2 \\ &= f'(x+h_1^*+h_2^*)L(h_1^*,h_2^*)/g'(h_1^*) \text{ (by A.7)} = \tilde{f}(x+h_2-h_2^*).\end{aligned}$$

In other words, if $h_2$ is close enough to $h_2^*$, or for all sufficiently small $\varepsilon$ in $\mathbb{R}$ and $x \geq 0$, $\tilde{f}(x)\tilde{f}(\varepsilon) = \tilde{f}(x+\varepsilon)$. That is, $\tilde{f}$ is a measurable function that satisfies the exponential law in a neighborhood of $x$ a.e., hence $\tilde{f}$ is an exponential function in that neighborhood (see, e.g. Aczél (1987)). Since $x$ can be chosen arbitrary a.e. this means that $\tilde{f}$ must be an exponential function of the form exp $(-\alpha x)$.

Therefore, $\frac{\partial}{\partial x}f(z,x)$ must have the form $v(z)\exp(-(z)x)$, and so without loss generality we can assume $f(z,x)$ has the form $f(z,x) = u(z)\exp(-\alpha(z)x) + \psi(z)$, where $u(z) \equiv -v(z)/\alpha(z)$. Since $f$ is a distribution function and decreasing in $x$, we can assume $u(z) \geq 0$, $\psi(z) \geq 0$, and $\alpha(z) > 0$ for every $z$. Since $f(z,\infty) = 0$, $\psi(z) = 0$ for any $z$. Since $f(0,x) = 0$ for any $x$, $u(0) = 0$. Since $f(M,0) = 1$, $u(M) + \psi(M) = 1$. Returning to the distributions $X$ and $Y$, $X \sim Y$ implies $f(z,a) - f(z,b) - f(0,a) + f(0,b) = f(M,c) - f(M,d) - f(z,c) + f(z,d)$ or $u(z)[\exp(-\alpha(z)a) - \exp(-\alpha(z)b) + \exp(-\alpha(z)c) - \exp(-\alpha(z)d)] = \exp(-\alpha(M)c) - \exp(-\alpha(M)d)$. And from Eq. (A.1) it follows that for all $t > 0$, $f(z,a+t) - f(z,b+t) - f(0,a+t) + f(0,b+t) = f(M,c+t) - f(M,d+t) - f(z,c+t) + f(z,d+t)$, or $\exp(-\alpha(z)t)u(z)[\exp(-\alpha(z)a) - \exp(-\alpha(z)b) + \exp(-\alpha(z)c) - \exp(-\alpha(z)d)] = \exp(-\alpha(M)t)[\exp(-\alpha(M)c) - \exp(-\alpha(M)d)]$. Hence $\exp(-\alpha(z)t) = \exp(-\alpha(M)t)$, or $\alpha(z) = \alpha(M)$ for all $z$ in [0, $M$]. Thus, $f(z,x) = u(z)\exp(-\alpha x) + \psi(z)$ for some $\alpha > 0$, which we wanted to show. □

## References


Aczél, J., 1987. A short course on functional equations. Dordrecht, Reidel.

Allais, M., 1988. The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function. In: B. Munier (Ed) Risk, Decision and Rationality. Dordrecht, Reidel.

Camerer, C., 1989. An experimental test of several generalized utility theories. Journal of Risk and Uncertainty 2, 61–104.

Carbone, E., Hey, J., 1994. An investigation into the determination of the error: A comparison of estimates of EU and non-EU preference functionals arising from pairwise choice experiments and complete ranking experiments. Mimeo, University of York.

Chateauneuf, A., 1990. On the use of comonotonicity in the axiomatization of EURDP theory for arbitrary consequences. Paper presented at the Fifth International Conference on the Foundations and Applications of Utility, Risk and Decision Theories, Duke University, North Carolina.

Chew, S.H., Epstein, L.G., Wakker, P.P., 1993. A unifying approach to axiomatic non-expected utility theories: Correction and comment. Journal of Economic Theory 59, 183–188.

Chew, S.H., Karni, E., Safra, Z., 1987. Risk aversion in the theory of expected utility with rank-dependent probabilities. Journal of Economic Theory 42, 370–381.

Diamond, P., Stiglitz, J., 1974. Increases in risk and in risk aversion. Journal of Economic Theory 8, 337–360.

Epstein Chew, S.H., Epstein, L.G., 1989. A unifying approach to axiomatic non-expected utility theories. Journal of Economic Theory 49, 207–240.

Feder, G., 1977. The impact of uncertainty in a class of objective functions. Journal of Economic Theory 16, 304–318.

Grant, S., Kajii, A., 1994. AUSI expected utility: An anticpated utility theory of relative disappointment aversion. CORE Discussion Paper No. 9445.

Green, J., Jullien, B., 1988. Ordinal independence in nonlinear utility theory. Journal of Risk and Uncertainty 1: 355–388. ('Erratum' 2 (1989): 119).

Gul, F., 1991. A theory of disappointment aversion. Econometrica 59, 667–686.

Hey, J.D., Orme, C., 1995. Investigating generalizations of expected utility theory using experimental data. Econometrica 62, 1291–1326.

Kahneman, D., Tversky, A., 1979. Prospect theory: an analysis of decision under risk. Econometrica 47, 263–291.

Karni, E., Safra, Z., 1987. 'Preference reversal' and the observability of preferences by experimental methods. Econometrica 55, 675–685.

Loomes, G., 1991. Evidence of a new violation of the independence axiom. Journal of Risk and Uncertainty 3, 91–108.

Luce, R.D., Fishburn, P.C., 1991. Rank- and sign-dependent linear utility models for finite first-order gambles. Journal of Risk and Uncertainty 4, 29–59.

Puppe, C., 1990. Preference functionals with prize-dependent distortions of probabilities. Economic Letters 33, 127–131.

Puppe, C., 1991. Distorted Probabilities and Choice Under Risk. Springer-Verlag Lecture Notes in Economics and Mathematical Systems #363.

Quiggin, J., 1982. A theory of anticipated utility. Journal of Economic Behaviour and Organizations 3, 323–343.

Quiggin, J., 1991. On the optimal design of lotteries. Economica 58, 1–16.

Quiggin, J., 1993. Generalized expected utility theory: the rank-dependent model. Kluwer Academic Publishers, London.

Quiggin, J., 1995. Economic choice in generalized expected utility theory AU. Theory and Decision 38(2), 153–171.

Royden, H.L., 1988. Real analysis. MacMillan, New York.

Rubinstein, A., 1988. Similarity and decision making under risk (Is there a utility theory resolution to the Allais paradox?). Journal of Economic Theory 46, 145–153.

Segal, U., 1987. Some remarks on Quiggin's anticipated utility. Journal of Economic Behavior and Organnizations 8, 145–154.

Segal, U., 1989. Anticipated utility: A measure representation approach. Annals of Operations Research 19, 359–373.

Segal, U., 1990. Two stage lotteries without the reduction axiom. Econometrica 58, 349–378.

Segal, U., 1993. The measure representation: A correction. Journal of Risk and Uncertainty 6, 99–107.

Starmer, C., 1992. Testing new theories of choice under uncertainty using the common consequence effect. Review of Economic Studies 59, 813–830.

Tversky, A., Kahneman, D., 1992. Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty 5, 297–323.

Wakker, P.P., 1993. Counter examples to Segal's measure representation theorem. Journal of Risk and Uncertainty 6, 91–98.

Wakker, P.P., 1994. Separating marginal utility and probabilistic risk aversion. Theory and Decision 36, 1–44.

Wu, G., 1994. An empirical test of ordinal independence. Journal of Risk and Uncertainty 9, 39–60.

Yaari, M.E., 1987. The dual theory of choice under risk. Econometrica 55, 95–115.