Many Good Choice Axioms: When Can Many-Good Lotteries Be Treated as Money Lotteries?*

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Without the Independence Axiom, a weaker substitution axiom, "ADI," is necessary for the preferences over money lotteries induced by the money metric utility function to be well behaved. Given ADI, the agent's preferences over many-good lotteries can be reconstructed from knowledge of preferences over money lotteries and over sure multivariate outcomes. Moreover, other substitution properties of underlying many-good lottery preferences are inherited by the money lottery preferences. We analyze the formal and intuitive nature of ADI and conclude that analysis of choice under uncertainty without ADI cannot be one dimensional.


1. INTRODUCTION AND MOTIVATION

Recent analyses of choice under uncertainty have been concerned almost exclusively with one dimensional risk—choices over "money lotteries." One could say that there are many good choice axioms, but few of these axioms deal with choices over many goods. This paper starts from the premise that agents' "primitive" preferences are not for money itself but are for the goods which money can buy. It is these underlying preferences over many-good lotteries that induce preferences over money lotteries. Two broad questions underlie what we have to say. First, what might we be implicitly assuming when we work in one dimension? Second, what

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1 For instance, the Rank Dependent Utility Theories of Quiggin [19], Yarri [24], and Green & Jullien [10], and the Generalized Expected Utility Theory of Machina [16].
properties of the underlying many-good lottery preferences can be inferred by studying the induced preferences over money lotteries, and how are these properties transmitted from the underlying to the induced preferences?

Discussions we have had with researchers in this area have led us to believe that most implicitly assume that preferences over money lotteries "represent" preferences over many-good lotteries, in some sense. What then is represented by a particular sum of money in a money lottery? Later we will have to define our terms exactly but, for purposes of motivation, let us sacrifice precision to be able to see the intuitive wood from the formal trees. One interpretation with intuitive appeal is that a given sum of money, \( m \), represents one of the commodity bundles that is most preferred in the budget set created by that sum of money and the given prices, \( p \). Let us call this bundle \( x^*(m, p) \). In this approach any probability mass on \( m \) in a simple money lottery should be thought of, roughly speaking, as representing an equivalent probability mass on \( x^* \) in a many-good lottery and, roughly speaking, preferences concerning the mass on \( x^* \) in a many-good lottery induce preferences concerning the mass on \( m \) in a money lottery.

The limitation of this approach is that we have captured only a "slice" of the underlying preferences. In this interpretation, the only many-good lottery preferences characterized by the money lottery preferences are those over lotteries whose support consists of points like \( x^* \), i.e., lotteries supported on the income expansion path induced by the prices \( p \). The risks of interest to economic agents, however, are not confined to uncertainty about incomes alone; for example, we might consider uncertainties about prices, rationing, legislation, and the weather. If the economics of choice under uncertainty wishes to be relevant on these issues, then we need to capture more of the underlying many-good lottery space than just an income expansion path.

It is tempting to think that if, in addition to prices and the agent's preferences over money lotteries, we also knew about his or her preferences over commodity bundles, then we would be able to infer his or her entire preference map over all many-good lotteries. As we shall see, this intuition contains some implicit assumptions, but where it holds true it offers a nice result. It says than an agent's entire preferences can be characterized by looking at two familiar sets of preferences: the first, the usual subject of choice under uncertainty, and the second, that of choice under certainty. We call this property the characterization result. Where it holds, the

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2 This concurs with Savage's [20, p. 86] interpretation of a prize in a lottery. We thank Mark Machina for this reference.
characterization result provides some justification for the focus on money lottery choice problems in the literature. Money lotteries can be said to constitute part of a "reduced form" of the many-good lottery problem. In this case, many goods can be usefully treated as one.

What interpretation of money lotteries will give us the characterization result, and what do such interpretations implicitly assume? Since we know that the interpretation of money lotteries outlined above is too narrow for our purposes, let us widen it as follows. A probability mass on the particular amount of money $m$ can be interpreted as representing a mass of that size on $x^*(m, p)$ as before, or a mass of that size on any commodity bundle which the agent regards to be equally as good as $x^*(m, p)$. In other words, the mass at $m$ could have been transformed from any commodity bundle in the particular indifference set that contains $x^*$. This transformation from the space of commodity bundles to money (and hence from many-good lotteries to money lotteries) is known as the money metric utility function.

In this paper we concentrate on the interpretation that money lottery preferences are induced from underlying preferences in a natural way using the money metric utility transformation. We believe that this interpretation is what most economists have in mind when they analyze money lottery preferences. We shall see that this interpretation is consistent with the characterization result, so we can claim to be capturing attitudes to much more than just income risk.

Provided we confine our attention to expected utility theory, the money metric interpretation of money lottery preferences is unproblematic. Perhaps it is due to our Independence-influenced intuitions that this interpretation seems so natural. But non-expected utility theories also work almost entirely with one dimensional lotteries. This suggests the following question: is rejecting Independence consistent with the money metric interpretation and the characterization result? In other words, can one dimensional non-expected utility theories claim to be applicable in contexts with more than just a very limited type of uncertainty?

The first task of this paper is to confront these questions. Section 2 establishes notation. Section 3 presents a necessary and sufficient restriction on the underlying preferences to ensure that preferences induced by the money metric utility transformation are well-behaved. This restriction is a substitution axiom that we call "ADI," the Axiom of Degenerate Inde-
pendence. ADI can be thought of as a restricted form of consequentialism. ADI allows us to characterize all welfare-relevant risk as uncertainty over the ranks of outcomes: an essentially one dimensional risk.

How restrictive is ADI? We show that ADI is consistent with the letter and at least part of the spirit of rejecting Independence. For example, we show that ADI is not violated by the Allais paradox and does not conflict with the introspective notion of the fear of “disappointment” that is often seen as “justifying” the choices made therein.

Section 4 shows that, with ADI, properties of the underlying preferences, such as Betweenness, are transmitted to and can be inferred from properties of the individual's induced preferences over money lotteries. A companion paper deals with the notions of risk and risk aversion in a many-good context without Independence and how these properties may be transmitted to and inferred from preferences over money lotteries.

Section 5 is more speculative. We consider the consequences of relaxing ADI. It could be argued that this is more genuinely in the spirit of rejecting Independence. We suggest that relaxing ADI is an appealing way to capture an intuitive notion we call intrinsic risk aversion. Relaxing ADI enables us to say that, even if two outcomes are indifferent under certainty, an agent might still prefer one outcome for sure to facing risk over which of the two he or she will receive.

Although this allows us much more generality, we should be aware of the downside of relaxing ADI. We would lose the characterization result and the convenient transmission of properties of preference over many-good lotteries to those over money lotteries. Although it is possible that we could replace ADI piecemeal, our view is that without ADI any analysis of choice under uncertainty will need to be explicitly multivariate. Many goods will no longer be able to be treated as one.

2. THE BASIC FRAMEWORK

Let \( X \) be the set of outcomes. We assume \( X \) is a compact metric space. The example of \( X = \prod_{i=1}^{n} X_i \), where, for each \( i \), \( X_i \) is a closed interval, is of special interest. In this case we refer to \( X \) as multi-commodity space and

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4 Roughly speaking, consequentialism says that an agent's choice of action should depend only on the (possibly risky) consequences of that action (see Hammond [11] for a more formal definition and its implications for decision theory). In a sense that can be made formal (see Karni and Schmeidler [14]), consequentialism says that the agent's preference for some sub-lottery \( f \) over some other sub-lottery \( g \) does not depend on the form or contents of the parent lottery in which \( f \) and \( g \) arise. ADI restricts this type of consequentialism to the case where the \( f \) and \( g \) in question are degenerate outcomes.

5 Grant, Kajii, and Polak [9].
to the outcomes as commodity bundles. Multi-commodity spaces are special in that they are endowed with a natural (although only partial) ordering. We will be careful in this paper to distinguish axioms that rely on the natural order of a space from those that rely on the order imposed on a space by the agent's preferences. This will be important, for example, when we come to consider stochastic dominance.

Let \( \mathcal{L}(X) \) denote the space of lotteries (probability distributions) over \( X \) endowed with the weak convergence topology. \( \mathcal{L}(X) \) is a separable, metrizable space. Let \( \delta_x \) denote the degenerate lottery on \( x \in X \). For \( x \in [0, 1] \), let \( xF \oplus (1-x)G \in \mathcal{L}(X) \) denote the probability mixture of \( F \) and \( G \).

We assume that the agent has preferences over \( \mathcal{L}(X) \) denoted by \( \succeq \). In the following analysis we take this preference relation as our primitive. We assume \( \succeq \) to be reflexive, transitive, complete, and continuous. Moreover, we assume that \( \succeq \) obey the Reduction of Compound Lottery Axiom (RCLA). Let \( \sim \) correspond to indifference and \( \succ \) to strict preference.

In this paper we shall also refer to other binary relations that describe the agent's preferences over particular spaces. We will call these binary relations "preferences" whether or not they satisfy the pre-order properties.

Given a non-empty subset \( \mathcal{A} \subseteq \mathcal{L}(X) \), we use \( \succeq_{\mathcal{A}} \) to denote the restriction of \( \succeq \) to \( \mathcal{A} \). With slight abuse of notation, we write \( \succeq_X \) to mean the agent's preferences over outcomes. Reflexivity, transitivity, completeness, and continuity\(^6\) of \( \succeq_X \) follow immediately from \( \succeq \) having these properties.

We call \( p(\cdot) \), a real-valued continuous function on \( X \), a price system. We regard \( p(x) \) as the money cost of outcome \( x \). When \( X \) is a multi-commodity space we assume \( p(\cdot) \) is a linear function, and (with slight abuse of notation) we denote \( p(x) = p \cdot x \). Let \( Y(p) \) be the income expansion path for a given price system \( p \); that is,

\[
Y(p) = \{y \in X : \exists x \in X \text{ s.t. } y \in \text{argmin } p(x) \text{ s.t. } x \succeq_x x \} \subseteq X.
\]

Thus \( \succeq_{Y(p)} \) refers to the restriction of \( \succeq \) on lotteries over that income expansion path.

3. INTERPRETING MANY AS ONE

Let \( M \) be a subset of \( \mathbb{R}_+ \). We call an element of \( \mathcal{L}(M) \), \( D \), a money lottery, and we denote preferences over money lotteries by \( \succeq_{\mathcal{L}(M)} \). Our first task is to clarify how money lottery preferences can characterize the

\(^6\) Indeed \( x \sim x \iff \delta_x \sim \delta_x \) in the weak topology, so continuity of \( \succeq \) implies the continuity of \( \succeq_X \).
many-good lottery preferences, $\succeq$, and what we mean when we say that $\succeq_{\mathcal{L}(M)}$ are induced by $\succeq$. As will become apparent these issues of money lottery preferences are flip sides of the same coin.

**Definition (Characterization).** We say that $\succeq_{\mathcal{L}(M)}$, for a given price system, characterizes $\succeq$ if knowledge of $\succeq_{\mathcal{L}(M)}$, $\succeq_{X}$, and the given price system are sufficient for us to reconstruct $\succeq$ in its entirety.

In this paper, we assume an agent has preferences over money lotteries only so far as these are induced from the underlying many-good lottery preferences. Formally,

**Definition (Induced Money Lottery Preferences).** Let $\mathcal{D} \subseteq \mathcal{L}(X)$, $\mathcal{D} \neq \emptyset$ and let $T : \mathcal{D} \rightarrow \mathcal{L}(M)$ be onto. The money lottery preferences, $\succeq_{\mathcal{L}(M)}$, induced by the transformation $T$ are given by: for all $\mathcal{D}, \mathcal{E} \in \mathcal{L}(M)$, if there exist $\mathcal{F}, \mathcal{G} \in \mathcal{D}$ such that $\mathcal{F} \in T^{-1}(\mathcal{D})$, $\mathcal{G} \in T^{-1}(\mathcal{E})$, and $\mathcal{F} \succeq_{\mathcal{D}} \mathcal{G}$ then $\mathcal{D} \succeq_{\mathcal{L}(M)} \mathcal{E}$.

Clearly $\succeq_{\mathcal{L}(M)}$ are reflexive and complete and, moreover, it is easy to show that $\succeq_{\mathcal{L}(M)}$ are continuous if $T$ is continuous. Note that, by the definition of $\succeq_{\mathcal{L}(M)}$, if $\mathcal{F} \succeq_{\mathcal{D}} \mathcal{G}$ then $T(\mathcal{F}) \succeq_{\mathcal{L}(M)} T(\mathcal{G})$. However, we would also like the induced preferences to be "consistent" with the underlying preferences in the sense that if $T(\mathcal{F}) \succeq_{\mathcal{L}(M)} T(\mathcal{G})$ then $\mathcal{F} \succeq_{\mathcal{D}} \mathcal{G}$. This "consistency" would ensure that no information is "lost" from $\succeq_{\mathcal{D}}$. For example, consistency ensures that $\mathcal{F} \succeq_{\mathcal{D}} \mathcal{G}$ if and only if $T(\mathcal{F}) \succeq_{\mathcal{L}(M)} T(\mathcal{G})$ (where $\succ_{\mathcal{L}(M)}$ is defined from $\succeq_{\mathcal{L}(M)}$ in the usual way), and that $\succeq_{\mathcal{L}(M)}$ are also transitive. To get this consistency property we require that the transformation $T$ satisfy the following condition:

**Condition C.** For all $\mathcal{F}, \mathcal{G} \in \mathcal{D}$, $T(\mathcal{F}) = T(\mathcal{G}) \Rightarrow \mathcal{F} \sim \mathcal{G}$.

**Lemma 3.1.** $T$ satisfies Condition C if and only if

$$T(\mathcal{F}) \succeq_{\mathcal{L}(M)} T(\mathcal{G}) \Rightarrow \mathcal{F} \succeq_{\mathcal{D}} \mathcal{G}.$$ 

**Proof.** See Appendix A.

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7 An alternative approach is to endow the agent both with preferences over many-good lotteries and with preferences over money lotteries and then to place some consistency requirement between the two sets of preferences. This second approach would change the definitions in this paper but leave most of the intuitions behind the results intact.

8 Note that $\mathcal{D}$ and/or $T$ can be chosen in such a way as to make the induced preferences $\succeq_{\mathcal{L}(M)}$ uninteresting; e.g., $\mathcal{F} \sim_{\mathcal{D}} \mathcal{G}$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{D}$. Also note that $\succeq_{\mathcal{L}(M)}$ may not be transitive.
Almost all analyses require that preferences respect *First Order Stochastic Dominance (FSD)*. Thus we might require that induced money lottery preferences should satisfy the following property (from Chew and Epstein [2]) which is equivalent to saying that \( \succeq_{\mathcal{X}(M)} \) respect FSD.

**Definition (Monotonicity M).** We say that \( \succeq_{\mathcal{X}(M)} \) exhibits monotonicity if:

\[
\forall x \in (0, 1], \forall m_1, m_2 \in \mathcal{M}, \forall D \in \mathcal{L}(M) \quad m_1 > m_2 \Rightarrow xD_{m_1} \oplus (1-x)D > xD_{m_2} \oplus (1-x)D.
\]

Note that the domain of the transformation, \( \mathcal{J} \), is not necessarily the entire space of many-good lotteries, \( \mathcal{L}(X) \). For example, consider the case where \( X \) is a multi-commodity space and \( \succeq_X \) is strictly convex. As in the introduction, let each sum of money be identified with the commodity bundle that the agent would choose to buy at given prices were he or she to win that sum of money. Formally, consider the bijection

\[
i^*_p : Y(p) \rightarrow M (\subseteq \mathbb{R}_+) \quad \text{s.t.} \quad \forall m \in \mathcal{M}, \quad i^*_p^{-1}(m) = x^*(m; p),
\]

where \( x^* \) is the Marshallian demand function, \( M \) is the range of \( i_p \), and \( Y(p) \) is the income expansion path for prices \( p \). Then we can define a transformation \( \hat{T}_p : \mathcal{L}(Y(p)) \rightarrow \mathcal{L}(M) \) as \( \hat{T}_p(F) = F \circ i^*_p^{-1} \) for \( F \) in \( \mathcal{L}(Y(p)) \).

If we confine our attention to this transformation, \( \hat{T}_p \), then Condition C is satisfied trivially without having to assume any further substitution axioms for the underlying many-good lottery preferences. However, there are disadvantages to restricting \( \mathcal{J} \) in this way. Without further substitution axioms, knowledge of the money lottery preferences provides no information about lotteries that include commodity bundles that are not elements of \( Y(p) \); i.e., \( \succeq_{\mathcal{X}(M)} \) is only characterizing the agent’s attitude to risks on this one dimensional subset of the multi-commodity space.

An alternative natural transformation from many-good lotteries to money lotteries, given a price system \( p \), is the money metric utility function. Formally, define a mapping \( m_p : X \rightarrow \mathbb{R} \) such that

\[
m_p(x) = \min_{\hat{x}} p(\hat{x}) \quad \text{s.t.} \quad \hat{x} \succeq_X x,
\]

and let \( M = \text{range of } m_p \). Then we can define a transformation

\[
T_p : \mathcal{L}(X) \rightarrow \mathcal{L}(M) \quad \text{s.t.} \quad \forall F \in \mathcal{L}(X), \quad T_p(F) = F \circ m_p^{-1}.
\]

This transformation has particular intuitive economic appeal. It labels each indifference set in \( X \) by the minimum amount of money it would take, given prices, to purchase a commodity bundle in that set. In this sense the

\[
F \circ i^*_p^{-1} \text{ a measure defined by: for any measurable set } K \subseteq M, \quad F \circ i^*_p^{-1}(K) = F(i^*_p(K)).
\]


transformed distribution over $\mathbf{M}$ and its pre-image in $\mathcal{L}(\mathbf{X})$ provide the same distributions over "sure" levels of satisfaction. We believe that this transformation is what most economists have in mind when they analyze preferences over money lotteries.

Interpreting money lottery preferences as having been induced by the money metric transformation places an important restriction on underlying preferences, $\succeq$. By identifying each outcome with its money metric equivalent, we are saying that an individual is indifferent between receiving a specified outcome $x \in \mathbf{X}$ or receiving an amount of money that, at the prevailing prices, would enable the agent to purchase an outcome "as good as" $x$. This intuition is captured by the following substitution axiom.

**Weak Axiom of Degenerate Independence (WADI).**

$$\forall F \in \mathcal{L}(\mathbf{X}), \forall \alpha \in (0, 1), \forall x, y \in \mathbf{X},$$

$$x \sim_x y \text{ implies } \alpha \delta_x \oplus (1 - \alpha)F \sim \alpha \delta_y \oplus (1 - \alpha)F.$$ 

We believe that most people who accept WADI will also accept the following strictly stronger axiom.

**Axiom of Degenerate Independence (ADI).**

$$\forall F \in \mathcal{L}(\mathbf{X}), \forall \alpha \in (0, 1) \forall x, y \in \mathbf{X},$$

$$x \succeq_x y \text{ if and only if } \alpha \delta_x \oplus (1 - \alpha)F \succeq \alpha \delta_y \oplus (1 - \alpha)F.\quad (10)$$

WADI ensures both that the money metric transformation satisfies Condition C and that the induced money lottery preferences completely characterize the underlying many-good lottery preferences. Moreover, with ADI, the induced money lottery preference relation satisfies Condition M (monotonicity). In fact, as the next proposition shows, the axioms are both necessary and sufficient.

**Proposition 3.2.** (a) $\succeq$ satisfies WADI if and only if the money metric transformation that maps $F \in \mathcal{L}(\mathbf{X})$ to $F \circ m_p^{-1} \in \mathcal{L}(\mathbf{M})$ satisfies Condition C. Furthermore, the induced preference relation, $\succeq_{\mathcal{L}(\mathbf{M})}$, inherits reflexivity, transitivity, completeness and continuity.

(b) $\succeq$ satisfies ADI if and only if $\succeq$ satisfies WADI and the preference relation, $\succeq_{\mathcal{L}(\mathbf{M})}$, induced by the money metric transformation, satisfies Condition M.

Preferences that obey ADI respect Fishburn and Vickson's [8] notion of non-dimensional first degree stochastic dominance. We use the name ADI to emphasize first, that we are referring to a property of preferences over lotteries, not a property of the lotteries themselves, and second, that this, like Independence, is a substitution axiom. We thank Moshe Buchinsky for bringing the name ADI to our attention.
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Proof. See Appendix A.

The intuition for Proposition 3.2(a) is as follows. WADI allows substitution of indifferent degenerate lotteries. Taken together with continuity, WADI enables us to concentrate the probability weight that a distribution assigns to an indifference set in outcome space, X, on one outcome in that indifference set; in particular, we could concentrate all the weight on the cheapest bundle in that set. Thus an agent would be indifferent between two simple distributions, F and G, if the probability weight assigned by F to each indifference set in X is the same as the probability weight assigned by G to that indifference set. This is precisely what is needed for \( F \circ m_p^{-1} \) to satisfy Condition C.

This axiom is also what is needed to achieve the characterization result. To see this first note that, given WADI, the two interpretations of money lottery preferences discussed in the introduction are equivalent. In particular, simple many-good lotteries can be reconstructed from simple money lotteries as follows. First, we can interpret any probability mass on \( m \), in the money lottery, as a similar mass on \( x^*(m; p) \), in outcome space. Then, using WADI, we can distribute the probability along the indifference set, preserving preference ranking. This procedure can be generalized as suggested by the following corollary.

**Corollary 3.2.1.** If \( \succeq \) satisfies WADI, then \( \succeq \) can be completely recovered from \( \succeq_{x(M)} \) and \( m_p : X \to M \); i.e., for all \( F, G \in \mathcal{L}(X) \),

\[
F \circ m_p^{-1} \succeq_{x(M)} G \circ m_p^{-1} \quad \text{iff} \quad F \succeq G. \quad (11)
\]

**Proof.** Follows directly from \( F \to F \circ m_p^{-1} \) satisfying Condition C and by construction of \( \succeq_{x(M)} \) as an induced preference relation.

Corollary 3.2.1 demonstrates that \( \succeq_{x(M)} \) and \( m_p : X \to M \) provide a convenient way to partition the information contained in a preference relation, \( \succeq \), that satisfies WADI. WADI allows us to treat lotteries over money as the reduced form of the many-good lottery problem. Given prices, if we know \( \succeq_{x(M)} \) and \( \succeq_x \), then we can infer the underlying preferences over arbitrary many-good lotteries. Moreover, we can infer many properties of the underlying preferences just from knowledge of properties of the induced money lottery preferences. Proposition 3.2(b) is an example of this; if \( \succeq_{x(M)} \) respects FSD, we can infer that \( \succeq \) satisfies ADI. Further examples are given in Section 4 below.

How restrictive are WADI and ADI? Since we believe it is unlikely

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(11) Provided \( \succeq \) satisfies WADI, \( m_p(\cdot) \) need not be restricted to the money metric. Corollary 3.1.2 applies equally to any utility representation of \( \succeq_x \).
that anyone who accepts WADI would reject ADI, we shall restrict our discussion to ADI.

To see what ADI implies, consider the two lotteries in Fig. 1. For ease of exposition, here and elsewhere, we have drawn these lotteries as two-stage lotteries, but recall that we are assuming RCLA.

ADI says that if the agent (weakly) prefers the sure commodity bundle containing three carrots and a peach to the bundle containing eight cherries and an apple, then he or she should (weakly) prefer lottery 1(a) to lottery 1(b). Where in the first lottery the agent got three carrots and a peach, in the second he or she now gets eight cherries and an apple, which are "worse (no better)."

The agent whose preferences obey ADI is not concerned about uncertainty over outcomes within the same indifference set. It is not surprising that this is the type of axiom we need to be able to work with one-dimensional lotteries. Once the outcome space has been partitioned into indifference sets, the set of indifference sets is isomorphic to the real line.

Although ADI is implied by Independence, and shares some of its consequentialist flavor, ADI is both formally and intuitively weaker than Independence.\(^\text{12}\) Consider the following examples using the lotteries

\[^{12}\text{However, many weakenings of Independence, e.g. Betweenness, do not imply (and are not implied by) ADI. See Example 4.2.}\]
illustrated in Fig. 2. Let $x, y, w,$ and $z$ be four commodity bundles like those in the previous example, i.e., let $\delta_x, \delta_y, \delta_w, \delta_z \in \mathcal{L}(X)$. Assume $\delta_x > \delta_w$ and $\delta_y > \delta_z$. If, for some $\alpha \in (0, 1)$, $\alpha \delta_x \oplus (1 - \alpha) \delta_y > \alpha \delta_w \oplus (1 - \alpha) \delta_z$, Independence would imply that lottery $2(a)$ is preferred to lottery $2(b)$. This is not implied by ADI.

The same figure can provide some intuition for the relation between ADI and Condition M set forth in Proposition 3.2(b). ADI does imply that lottery $2(c)$ is preferred to lottery $2(d)$. To see this, first replace $\delta_z$ by $\delta_y$ in lottery $2(c)$ to form lottery $2(a)$. Then replace $\delta_x$ by $\delta_w$ to form lottery $2(d)$. Hence by the transitivity of "$>$" and the repeated application of ADI, the first lottery is preferred to the third. The key point to note is that ADI applies to multiple degenerate substitutions if they are all in the same "direction," i.e., monotonic with respect to preference.

How might we "rationalize" an agent's preferences satisfying ADI but violating Independence? Roughly speaking, failures of Independence can

Fig. 2. If $x>y$ and $z>x$, then by ADI lottery 2(c) is preferred to lottery 2(d), but ADI says nothing about the preference between lotteries 2(a) and 2(b).
occur when the agent's attitude to the risk of getting a good or bad outcome in substitute sublotteries depends on what else is in the parent lottery. The classic example of this is the lotteries in the Allais Paradox. (Refer to Fig. 3.)

Many agents are not willing to bear the very small risk of getting $0 in the sublottery in lottery 3(b) where the lower branch has $1 million. Hence lottery 3(a) > lottery 3(b). But they are prepared to bear this risk where the lower branch has $0. Hence lottery 3(d) > lottery 3(c). Many rationalize these preferences by saying they could not live with the disappointment if the worst outcome in lottery 3(b) occurred, since they would have foregone the opportunity of receiving $1 million for sure. These preferences violate Independence but do not violate ADI. ADI limits us to substituting only degenerate sublotteries and therefore has no "bite" if new risks of winning

![Lottery Diagram]

FIG. 3. The four lotteries here constitute the well-known Allais Paradox. Many people exhibit a preference for lottery 3(a) over 3(b) but prefer lottery 3(d) to 3(c). Such preferences violate Independence.
and losing are introduced. With the substitutions that ADI applies to, there are no new possibilities of being disappointed. The only restriction that ADI places on the shape of indifference curves in the probability simplex is that it forces them to be "upward sloping."

Despite the close relationship between ADI in $\succeq$ and Condition M in $\succeq_{x(M)}$, ADI is not the only way to extend Condition M to a multi-commodity space. The following is a natural extension:

**Axiom of Many Goods Monotonicity (MGM).**

$$\forall x, y \in X, \forall F \in \mathcal{L}(X), \forall \alpha \in (0, 1]$$

$$[x \succeq y, x \neq y] \implies [x\delta_x \oplus (1-\alpha)F \succ x\delta_y \oplus (1-\alpha)F].$$

Note that, unlike MGM, the definition of ADI does not depend upon any intrinsic ordering of the outcome space, $X$, prior to the application of preferences. Hence, ADI is well defined when $X$ is not a multi-commodity space. But even when $X$ does have an intrinsic ordering and when $\succeq_{x(X)}$, respect that ordering, MGM may still not be equivalent to ADI. The formal relationship between ADI and MGM is given in the following proposition.

**Proposition 3.3.** Given $\succeq_x$ are strictly monotonic in $X$,

$WADI$ and $MGM \Leftrightarrow ADI$.

**Proof.** "\(\Rightarrow\)" follows trivially from the definition of ADI and strict monotonicity.

"\(\Leftarrow\)" Choose $x, z \in X$ such that $x \succ x z$. We can find $y$ such that $y \geq z, y \neq z$ and such that $y \sim_x x$. For any $F \in \mathcal{L}(X)$ and $\alpha \in (0, 1)$, MGM implies $(1-\alpha)F \oplus \delta_y \succ (1-\alpha)F \oplus \delta_x$ and hence WADI implies $(1-\alpha)F \oplus \delta_x \succ (1-\alpha)F \oplus \delta_z$. That is, ADI holds.

MGM on its own would not be sufficient to imply ADI since, without WADI, the last step of the previous argument would not go through in general. Recall that if $\succeq$ did not satisfy WADI we can still induce money lottery preferences, $\succeq_{x(M)}$, directly from the preferences for lotteries over an income expansion path, $\succeq_{x(Y(P))}$, provided each quantity of money corresponds to exactly one consumption bundle. However, if we do not assume WADI, preferences over commodity bundle lotteries may obey

13. Just as M implies that preferences respect FSD in one dimension, one can verify that MGM implies strict preference between two distributions that satisfy the Levhari, Paroush, and Peleg's [15] generalization of FSD for multivariate distributions.

14. Recall that ADI is analogous to Fishburn and Vickson's [8] notion of non-dimensional first degree stochastic dominance. See note 10 above.
multi-dimensional first order stochastic dominance (i.e., MGM) and yet the induced preferences over money lotteries may fail to obey one dimensional first order stochastic dominance (i.e., Condition M).

To see this consider the following example. Let $X = \{(2,1), (1,2), (3,3)\}$. By MGM, we know that $(3,3)$ is preferred to the other two bundles. Let us assume that $(1,2) >_X (2,1)$. Let $M = \{m_1, m_2, m_3\}$ such that $m_1$ is the cost of $(2,1)$, $m_2$ is the cost of $(1,2)$, and $m_3$ is the cost of $(3,3)$, and $m_1 < m_2 < m_3$. The preferences over $\mathcal{L}(X)$ shown in Fig. 4 obey MGM but do not obey ADI. Note that $\delta_{(2,1)}$ is strictly preferred to $F$, where $F = \frac{1}{3}\delta_{(1,2)} \oplus \frac{2}{3}\delta_{(2,1)}$ and hence $\delta_{m_1} \succ \mathcal{L}(M) \frac{1}{3}\delta_{m_2} \oplus \frac{2}{3}\delta_{m_1}$, which violates Condition M. Note that MGM ensures that moving probability weight from a commodity bundle to another which has at least as much of each good and more of at least one good is viewed as a favorable change by the agent. However, to ensure that Condition M is satisfied for the induced preferences over money lotteries, we require movement of probability weight up an income expansion path (IEP) always to be desirable for the agent. Thus, in cases where the direction of the IEP is not increasing in terms of the partial ordering of the commodity space (i.e., in the presence of an inferior good) a contradiction may arise, as the example exhibits.

---

**Fig. 4.** Points in the above Machina–Marshak simplex represent both two-good lotteries for the three commodity bundle outcome space $\{(1,2), (2,1), (3,3)\}$ and money lotteries over $\{m_1, m_2, m_3\}$. A point’s $x$-coordinate represents the probability assigned to commodity bundle $(2,1)$ (resp. $m_1$) and its $y$-coordinate represents the probability assigned to commodity bundle $(3,3)$. The preference map drawn provides an example of preferences which satisfy MGM but do not satisfy ADI and so the induced money lottery preferences do not exhibit Condition M. Note $\delta_{(2,1)}$ is strictly preferred to $F$ where $F = \frac{1}{3}\delta_{(1,2)} \oplus \frac{2}{3}\delta_{(2,1)}$ and hence $\delta_{m_1} \succ \mathcal{L}(M) \frac{1}{3}\delta_{m_2} \oplus \frac{2}{3}\delta_{m_1}$. 

Nothing hinges on the outcome space being discrete, and the interested reader is referred to Appendix B where we provide an example with a continuously divisible commodity space.

4. Non-Expected Utility Theories in a Many-Good Context

In the last section we showed that given WADI we can recover preferences over arbitrary many-good lotteries once we know preferences over money lotteries and preferences over sure outcomes. Moreover, given WADI, knowledge that preferences over money lotteries exhibit Condition M is sufficient to infer that preferences over many-good lotteries obey ADI (and MGM if X is a multi-commodity space).

Throughout this section we assume ADI. We ask: what can be inferred about the underlying preferences over many-good lotteries from knowledge that induced preferences over money lotteries obey substitution axioms familiar from Non-Expected Utility theories, such as Betweenness? Alternatively: what do we have to assume about the underlying preferences in order to ensure that the induced money lottery preferences satisfy a particular substitution axiom?

We show that substitution axioms, such as Betweenness, imposed on the primitive preference ordering, \( \succeq \), are inherited by the induced preference relation, \( \succeq_{\mathcal{L}(M)} \). The intuition for this is as follows. Let \( W \) be a continuous utility function representation for \( \succeq_{\mathcal{L}(M)} \). Then there is a natural utility function, \( V \), representing \( \succeq \), where for all \( F \in \mathcal{L}(X) \), \( V(F) = W(F \circ m_p^{-1}) \).

Thus, if the form of the function \( W \) is known, we can immediately characterize the form of \( V \). Note that the money metric transformation, \( T_p(F) = F \circ m_p^{-1} \), operates linearly on \( \mathcal{L}(X) \). Therefore, “substitution axioms” for \( \succeq \), in addition to ADI, will be inherited by the induced preference relation \( \succeq_{\mathcal{L}(M)} \), since these “substitution axioms” refer to the partial separability of \( \succeq \).

Let us formalize this intuition. Many of the axioms from Non-Expected Utility Theories can be seen as elements of the following class of substitution axioms.

**General Substitution Axiom (GSA).** Let \( S \subseteq \mathcal{L}(X) \times \mathcal{L}(X) \times \mathcal{L}(X) \times (0, 1) \).

\( \succeq \) satisfies GSA with respect to \( S \) if, for all \( (F, G, H, \alpha) \in S \), \( \alpha F \oplus (1 - \alpha) H \succeq \alpha G \oplus (1 - \alpha) H \).

Debreu [5] shows that a continuous, complete pre-order defined on a separable, metrizable space, such as \( \succeq \), can be represented by a continuous utility function. Thus, by Proposition 3.2, if \( \succeq \) satisfy ADI, then \( \succeq_{\mathcal{L}(M)} \) also has a utility function representation.
To see how well-known substitution axioms fit into this framework, consider the following sets:

\[ S_D = \{ (F, G, H, \alpha) : F, G \text{ degenerate, } F \succeq G \} \]
\[ S_B = \{ (F, G, H, \alpha) : F \succeq G, (H = F \text{ or } H = G) \} \]
\[ S_I = \{ (F, G, H, \alpha) : F \succeq G \} \]
\[ S_O = \{ (F, G, H, \alpha) : \exists x \in X \text{ s.t. } \text{supp}(F), \text{supp}(G) \subseteq U(x); \text{supp}(H) \subseteq L(x); \text{and } \exists K, \text{supp}(K) \subseteq L(x) \} \]

where \( L(x) = \{ z \in X : x \succ z \} \) and \( U(x) = \{ z \in X : z \succeq x \} \) and \( \text{supp}(F) \) stands for the support of \( F \).

Preferences over many-goods satisfying GSA with respect to \( S_D \) corresponds to ADI; \( S_B \) corresponds to Betweenness; \( S_I \) corresponds to Independence; \( S_O \) corresponds to Ordinal Independence.

The formal answers to the two questions we posed at the beginning of this section are given by the following proposition.

**Proposition 4.1.** Suppose the underlying preference ordering, \( \succeq \), satisfies ADI. If \( \succeq \) satisfies GSA with respect to \( S \subseteq \mathcal{L}(X) \times \mathcal{L}(X) \times \mathcal{L}(X) \times (0, 1) \), then \( \succeq_{\mathcal{L}(M)} \) (induced by the money metric transformation, \( T_{\rho}(F) = F \circ m^{-1} \)) satisfies GSA with respect to \( R(S) = \{ (T_{\rho}(F), T_{\rho}(G), T_{\rho}(H), \alpha) : (F, G, H, \alpha) \in S \} \).

Conversely, if \( \succeq_{\mathcal{L}(M)} \) satisfies GSA with respect to \( R \subseteq \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(M) \times (0, 1) \), then \( \succeq \) satisfies GSA with respect to \( S(R) = \{ (F, G, H, \alpha) : (T_{\rho}(F), T_{\rho}(G), T_{\rho}(H), \alpha) \in R \} \).

\[ ^{15} \text{We believe this is a natural many-good extension of Green & Jullien's } [10] \text{ Ordinal Independence. Intuitively GSA with respect to } S_O \text{ can be viewed as an editing axiom. If two distributions coincide above an outcome indifference curve, then the preference between those two distributions is determined by their restriction to those outcomes on which they differ. Segal } [21] \text{ extends his treatment of rank-dependent expected utility to a commodity space in an analogous manner.} \]
Proof. See Appendix A.

For example if $\succsim$ satisfies ADI and Betweenness (Independence, Ordinal Independence, etc.) then so does the preference relation, $\succsim_{\mathcal{L}(M)}$, induced by $T_{\rho}(F) = F \circ m_{\rho}^{-1}$.

Now it is straightforward to characterize utility representations for $\succsim$. For instance, suppose $\succsim$ satisfies ADI and Betweenness. Thus as Dekel [4] proved $\succsim_{\mathcal{L}(M)}$ can be represented by the continuous utility function $W(\cdot)$ which is implicitly defined as:

$$\int_{\mathcal{D}} \phi(m', W(D)) \, dD(m') = 0, \quad \phi_1 > 0, \quad \phi_2 < 0 \quad \forall D \in \mathcal{L}(M).$$

Hence we have $V(F) = W(F \circ m_{\rho}^{-1})$ as the characterization of the utility function representation for $\succsim$. The representations for other substitution axioms can be similarly derived.

Thus if $\succsim$ satisfies ADI, we can analyze substitution properties of $\succsim$ indirectly by working with the induced money lottery preferences, $\succsim_{\mathcal{L}(M)}$, since it inherits these properties. Without ADI, however, we would not have the characterization result and thus would have to work directly in many-good lottery space.

It does not follow that, because $\succsim$ satisfies some substitution axiom, it automatically satisfies ADI. For example, consider Betweenness. Betweenness places restrictions on preferences when probability weight is moved between two given distributions. ADI, on the other hand, places restriction on preferences when a probability mass from a distribution is moved to a better outcome. The following counter-example demonstrates that $\succsim$ satisfying Betweenness does not imply that it also satisfies ADI.

**Example 4.2.** Consider $V(F)$, the functional representation of many good preferences implicitly defined by

$$\int_{x} \frac{(u(x) - V(F))}{x_1} \, dF(x) = 0.$$

(Assume that for all $x \in X, x_1 > 0$), where $u(x)$ is a functional representation of the preferences over outcomes. It is easy to check that $V(F)$ is well defined. It follows immediately that if $V(F) \geq V(G)$ then $V(F) \geq$
Thus these preferences satisfy Betweenness. It also follows that for these preferences, \( x \succeq x y \) if and only if \( u(x) \geq u(y) \). So choose \( x, y, z \in X \) such that \( u = u(x) = u(y) \neq u(z) \) and \( x_1 \neq y_1, z_1 = 1 \). ADI then implies that, for any \( \alpha \in (0, 1) \)

\[
V(\alpha \delta_x \oplus (1 - \alpha) \delta_y) = V(\alpha \delta_y \oplus (1 - \alpha) \delta_x) = V_x.
\]

Applying the implicit definition for \( V(F) \), \( V_x \) must satisfy

\[
\alpha \left[ \frac{V_x - u}{x_1} \right] + (1 - \alpha) \left[ V_x - u(z) \right] = 0 \tag{4.1}
\]

and

\[
\alpha \left[ \frac{V_x - u}{y_1} \right] + (1 - \alpha) \left[ V_x - u(z) \right] = 0. \tag{4.2}
\]

But no \( V_x \) satisfies both (4.1) and (4.2) because they are linearly independent (since \( x_1 \neq y_1 \)). Hence these preferences obey Betweenness but do not satisfy ADI.

5. LIFE WITHOUT ADI

The results of the sections above have shown that if the underlying preferences over many-good lotteries, \( \succsim \), satisfy ADI then certain other properties of \( \succsim \) are inherited in a natural way by the induced preferences over money lotteries. Furthermore, with ADI, we can make inferences about the underlying preferences from knowledge of certain properties of the money lottery preferences. Without ADI (strictly speaking without WADI), it is no longer clear that money lottery preferences are informative of underlying preferences, except, tautologically, for lotteries whose supports are confined to a specific income expansion path in outcome space. Thus life without ADI may prove to be uncomfortable.

Why might we want to consider preferences that violate ADI? We offer three reasons, the first of which is relatively new to economics. The other two reasons are often cited as arguments for dropping Independence and we shall argue that they apply equally to ADI.

Conventional economic theory only considers risk in so far as there is uncertainty about which indifference sets of outcomes may be achieved. Recall that, with ADI, if an agent is indifferent between an orange and an apple then she or he will also be indifferent between an orange for sure and a coin toss between an orange and an apple. But it is possible to imagine an agent who does not like this uncertainty per se. Such an agent may
prefer, *ex ante*, one outcome for sure, even though, after the uncertainty is resolved, the agent is indifferent as to which piece of fruit she or he might receive. We refer to this as *intrinsic* risk aversion.¹⁹

Note that intrinsic risk aversion cannot arise for the case of lotteries over one good, since with one good, indifference sets of outcomes have only a single element.²⁰ Thus, if the agent exhibits intrinsic risk aversion, we should not expect to be able to infer his or her underlying preferences over arbitrary many-good lotteries from knowledge of his or her preferences over outcomes and over money lotteries.

Secondly, relaxing ADI takes us away from consequentialism and allows the individual to care about the process by which outcomes are allocated. For instance, recall Machina's [17] example of a mom who has an indivisible treat which she can give to either her daughter, Abigail, or her son, Benjie. We are told she is indifferent between the outcome of Abigail or Benjie receiving the treat but prefers the (50:50) lottery to all other lotteries that allocate the treat to either child. If we view Abigail getting the treat and Benjie getting the treat as two distinct goods, then we can view mom's preferences for these lotteries as violating ADI.

Mom could be viewed as an intrinsic risk lover in this example, but a more likely interpretation is that she regards the (50:50) lottery as the "fairest" procedure for allocating the treat. This procedural consideration is completely absent in the consequentialist reasoning behind ADI.

A third reason why we might want to drop ADI is that often the choice problem in question is in fact a reduced form of a dynamic choice problem. For example, Ben might be indifferent between taking next year's vacation in Australia or Japan, but since he wants to read beforehand about the history of the country he visits, he will prefer to know his destination for sure to having a lottery performed at the airport. A static version of this problem fails to capture the time element, and hence Ben's preferences might appear to violate ADI.²¹

Some of the features of underlying preferences not satisfying ADI are similar to Segal's approach of relaxing the reduction of compound lotteries axiom (RCLA) for preferences over lotteries of one good.²² Violations of RCLA might occur when an agent cares about which point in a lottery tree the uncertainty is resolved. As with the many-good analysis, relaxing

¹⁹ Our notion of intrinsic risk aversion is not the same as that discussed by Dyer and Sarin [7]. They "label" indifference sets in outcome space by a level of riskless cardinal utility, whereas we "label" by money. The key difference, however, is that they nowhere relax ADI; in fact, they assume Independence throughout.

²⁰ Assuming that preferences are strictly monotonic in the one good.

²¹ This corresponds to a standard argument for dropping Independence for lottery preferences, see for example, Dreze and Modigliani [6] or Machina [17].

²² See Segal [22]. We thank Jerry Green for suggesting this similarity to us.
RCLA enables the agent to prefer one lottery to another, even though the agent is indifferent between the outcomes of the two lotteries.

Both violations of ADI and violations of RCLA can be interpreted as examples of either intrinsic risk aversion or reduced forms of dynamic choice problems. In using introspection to develop intuitions for either case, however, it is hard to disentangle the static from the dynamic problem. It could be argued that what we do not like is having to live with intrinsic risk over time.

It is worth noting that, if \( X \) is itself a set of lotteries over some outcome space \( S \), then ADI, defined for the preferences over \( \mathcal{L}(X) \), together with RCLA implies Independence for the preferences over the lottery space \( X \). This follows immediately from Segal’s [22] Theorem 3. In this context ADI corresponds to Segal’s Compound Independence Axiom. However, we would argue that, if \( X \) is the space of lotteries over \( S \), then preferences over \( \mathcal{L}(X) \) should not be regarded as the agent’s primitive underlying preferences. Rather, the agent’s primitive preferences should be over \( \mathcal{L}(S) \). Hence ADI should only apply to substitutions of degenerate elements of \( \mathcal{L}(S) \).

If we wish to analyze intrinsic risk aversion, there may be advantages in maintaining RCLA but dropping ADI. If one relaxes RCLA, one can arbitrarily expand the space of lotteries by increasing the number of stages in which lotteries can be expressed. By relaxing ADI, but maintaining RCLA, the space of lotteries that one needs to consider cannot become bigger than \( \mathcal{L}(X) \). We conjecture that the route of relaxing ADI but still employing RCLA may prove more tractable.

A fully fledged analysis of preferences that do not satisfy ADI is beyond the scope of this paper, but the following observations may be instructive as to likely issues such an analysis might address. All of the following issues are possible directions for further research.

If \( X \) is a commodity space, \( \preceq_X \) are monotonic, but preferences do not satisfy ADI, then it is quite conceivable that they do not satisfy MGM either. This violation of “monotonicity” can result from the agent being willing to trade off some probabilistic gains in desirability of consumption bundles for surety of a specific bundle.

Dropping ADI allows us to consider price as well as income risk aversion. In fact, the degree of price risk and price risk aversion could be used as a gauge of intrinsic risk and a test for intrinsic risk aversion. Consider some \( F \in \mathcal{L}(X) \). Roughly speaking (for \( \preceq_X \) smooth and strictly convex), the marginal distribution of \( F \) over indifference sets of commodity bundles can be thought of as the real income risk of the distribution, while the

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23 Segal avoids this problem by explicitly dealing with only two stage lotteries, but even with this restriction the analysis becomes complicated.
marginal distribution of $F$ over different income expansion paths, given some metric, might measure the *price* risk of the distribution.

Section 4 showed that ADI is formally weaker than Independence and that some form of consequentialism could "rationalize" preferences that satisfied one but not the other. However, we have argued in this section that many introspective objections to Independence apply equally to ADI. Many theorists, who find Independence unacceptable, are also likely to reject ADI. This paper leads us to believe, however, that if non-expected utility theory wishes to investigate the consequences of relaxing ADI, in a context broader than just income risk, then that investigation should not be confined to analysis of one dimensional lotteries. Life without ADI forces choice under uncertainty away from a one dimensional world.

**APPENDIX A**

**Proof of Lemma 3.1.** "$\Rightarrow$" For all $(T(F), T(G)) \in \succeq_{\mathcal{M}}$, there exists $F' \in T^{-1}(T(F))$ and $G' \in T^{-1}(T(G))$ such that $F' \succeq_{\mathcal{M}} G'$. By Condition C, $F' \sim_{\mathcal{M}} F$ and $G' \sim_{\mathcal{M}} G$. Hence by transitivity $F \succeq_{\mathcal{M}} G$. 

"$\Leftarrow$" For all $F, G \in \mathcal{M}$, $T(F) \succeq_{\mathcal{M}} T(G)$ implies $F \succeq_{\mathcal{M}} G$. In particular, if $T(F) = T(G)$, we have $F \succeq_{\mathcal{M}} G$ and $G \succeq_{\mathcal{M}} F$; i.e., $F \sim_{\mathcal{M}} G$. 

**Proof of Proposition 3.2.** Proof of 3.2(a). It can be shown by applying the measurable selection theorem (see Hildenbrand [12]) that there exists a measurable set $A$ in $X$ such that $A \cap m^{-1}_p(r)$ is a singleton for any $r \in \mathcal{M}$. We regard the set of probability distributions over $A$, $\mathcal{L}(A)$, naturally as a subset of $\mathcal{L}(X)$.

It is easy to see that, since $m$ induces a bijection between $A$ and $\mathcal{M}$, for all $\mu \in \mathcal{L}(\mathcal{M})$, there exists a unique $F \in \mathcal{L}(A) \subseteq \mathcal{L}(X)$ s.t. $F \circ m^{-1} = \mu$. As a corollary, we have $F \mapsto F \circ m^{-1}$ is onto.

Now we show that Condition C for $F \circ m^{-1} \Rightarrow$ WADI. Recall consistency requires $F \circ m^{-1} = G \circ m^{-1}$ only if $F \sim G$. Choose any $x, y \in X$, $H \in \mathcal{L}(X)$ and $\alpha \in [0, 1]$, and let $F = \alpha \delta_x \oplus (1 - \alpha)H$ and $G = \alpha \delta_y \oplus (1 - \alpha)H$, then $T_p(F) = \alpha \delta_{m_p(x)} \oplus (1 - \alpha)H \circ m^{-1}_p$ and $T_p(G) = \alpha \delta_{m_p(y)} \oplus (1 - \alpha)H \circ m^{-1}_p$. Thus $T_p(F) = T_p(G)$ implies $m_p(x) = m_p(y) \Leftrightarrow \delta_x \sim \delta_y$, therefore $\alpha \delta_x \oplus (1 - \alpha)H \sim \alpha \delta_y \oplus (1 - \alpha)H$. 

WADI $\Rightarrow$ Condition C for $F \circ m^{-1}$. We shall show this for simple first, and use a standard continuity argument. Consider $F = \sum_{i=1}^J \alpha_i \delta_{x_i}$ with $\alpha_i \geq 0$ and $\sum_{i=1}^J \alpha_i = 1$. WADI implies if $x_k \sim x_1$, then $F' = \sum_{j \neq k, j \neq 1} \alpha_j \delta_{x_j} \oplus (\alpha_k + \alpha_1) \delta_{x_k} \sim F$. In general, if $x_{ij} \sim x_{i1}$ for all $i = 1, \ldots, l$
Let \( j = 1, \ldots, J(i) \) and \( \alpha_{ij} \geq 0, \sum_{j=1}^{J(i)} \alpha_{ij} = 1 \), then by a finite repetition of the above operation we can show that

\[
\mathcal{S} = \sum_{i=1}^{I} \sum_{j=1}^{J(i)} \alpha_{ij} \delta_{x_{ij}} \sim \mathcal{S}_l \equiv \sum_{i=1}^{I} \left( \sum_{j=1}^{J(i)} \alpha_{ij} \right) \delta_{x_{i1}}.
\]

But

\[
\mathcal{S} \cdot m_p^{-1} = \sum_{i=1}^{I} \sum_{j=1}^{J(i)} \alpha_{ij} \delta_{m_p(x_{ij})}
\]

\[
= \sum_{i=1}^{I} \left( \sum_{j=1}^{J(i)} \alpha_{ij} \right) \delta_{m_p(x_{i1})} = \mathcal{S}_l \cdot m_p^{-1}.
\]

Since \( m_p(x_{ij}) = m_p(x_{i1}) \) \( \forall j = 1, \ldots, J(i) \) \( i = 1, \ldots, I \), thus WADI implies consistency for the set of simple lotteries over \( X \). Since \( F \mapsto F \cdot m_p^{-1} \) and \( V \) are continuous and the simple lotteries are dense in \( \mathcal{L}(X) \), we can now show that consistency is implied for all lotteries in \( \mathcal{L}(X) \).

Let \( F \cdot m_p^{-1} = G \cdot m_p^{-1} = \mu \). We want to show \( F \sim G \). Now consider a sequence of subsets \( \{X^n : n = 1, \ldots\} \) of \( X \),

\[
X^n = \{x^n_{ij} : i = 1, \ldots, I(n), j = 1, \ldots, J(n)\}, \text{ such that}
\]

(i) \( (\forall n) X^n \subseteq X^{n+1} \)

(ii) \( m_p(x^n_{i1}) = m_p(x^n_{j1}) \) for all \( i = 1, \ldots, I(n), j = 1, \ldots, J(n) \)

(iii) \( \bigcup_{n=1}^{\infty} X^n \) is dense in \( X \).

Associating with \( \{X^n\} \), we can construct a sequence of partitions \( \{\mathcal{P}^n\} \) of \( X \), \( \mathcal{P}^n = \{E^n_{ij} \subseteq X, E^n_{ij} \text{ is measurable, } i = 1, \ldots, I(n), j = 1, \ldots, J(n)\} \)

(iv) \( x_{kl} \in E^n_{ij} \leftrightarrow i = k, j = l \)

(v) \( m_p(E^n_{ij}) = m_p(E^n_{i1}) \) for all \( i \) and \( j = 1, \ldots, J(n) \).

Define

\[
F_n = \sum_{i,j} \alpha_{ij}^n \cdot \delta_{x_{ij}^n}, \quad \text{where} \quad \alpha_{ij}^n = F(E^n_{ij})
\]

\[
G_n = \sum_{i,j} \beta_{ij}^n \cdot \delta_{x_{ij}^n}, \quad \text{where} \quad \beta_{ij}^n = G(E^n_{ij}).
\]

By construction \( \sum_i \alpha_{ij}^n = \sum_j \beta_{ij}^n \), so \( F_n \sim G_n \) by WADI

Claim. \( F_n \rightarrow F \) weakly. That is, for any continuous function \( f \) on \( X \),

\[\lim_n \int_X f \, dF_n = \lim_n [\sum_{i,j} f(x_{ij}^n) \, F(E^n_{ij})].\]
This claim holds as the RHS is indeed the Riemann integral of \( f \) with respect to \( F \), which coincides with the Lebesgue integral of \( f \) with respect to \( F \) since \( f \) is continuous.

Reflexivity, transitivity, and completeness of \( \succeq_{\mathcal{L}(M)} \) follow immediately from the construction of \( \succeq_{\mathcal{L}(M)} \). Continuity follows from the fact that \( F \mapsto F \circ m^{-1} \) is continuous (see, for example, Hildenbrand [12]) and so it is a closed mapping because \( \mathcal{L}(X) \) is compact.

Proof of 3.2(b). This follows immediately since by construction we have

\[
\forall x \in (0, 1], \forall D \in \mathcal{L}(M), \forall m_1, m_2 \in M
\]

\[
[\alpha \delta_{m_1} \oplus (1 - \alpha) D \succ_{\mathcal{L}(M)} \alpha \delta_{m_2} \oplus (1 - \alpha) D] \Leftrightarrow [\alpha \delta_x \oplus (1 - \alpha) F \succ \alpha \delta_y \oplus (1 - \alpha) F]
\]

\[
\forall x \in m_p^{-1}(m_1), \forall y \in m_p^{-1}(m_2) \text{ and } \forall F \in T_{m_p}^{-1}(D).
\]

(Note, if extra income does not put the agent on a higher indifference curve, then it is not in the range of \( m_p \) and thus is not in \( M \).)

Proof of Proposition 4.1. Assume \( \succeq \) satisfies GSA with respect to \( S \). Choose any \( (F \circ m_p^{-1}, G \circ m_p^{-1}, H \circ m_p^{-1}, \alpha) \in \mathbb{R}(S) \subseteq \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(M) \times (0, 1) \). Since \( (F, G, H, \alpha) \in S \), it follows \( \alpha F \oplus (1 - \alpha) H \succeq \alpha G \oplus (1 - \alpha) H \). Now by the construction of \( \succeq_{\mathcal{L}(M)} \),

\[
[\alpha F \oplus (1 - \alpha) H] \circ m_p^{-1} \succeq_{\mathcal{L}(M)} [\alpha G \oplus (1 - \alpha) H] \circ m_p^{-1}.
\]

Thus the linearity of the money metric transformation implies

\[
\alpha F \circ m_p^{-1} \oplus (1 - \alpha) H \circ m_p^{-1} \succeq_{\mathcal{L}(M)} \alpha G \circ m_p^{-1} \oplus (1 - \alpha) H \circ m_p^{-1}
\]

i.e., \( \succeq_{\mathcal{L}(M)} \) satisfies GSA with respect to \( \mathbb{R}(S) \).

Next assume \( \succeq_{\mathcal{L}(M)} \) satisfies GSA with respect to \( \mathbb{R} \). Choose any \( (F, G, H, \alpha) \) such that \( (F \circ m_p^{-1}, G \circ m_p^{-1}, H \circ m_p^{-1}, \alpha) \in \mathbb{R} \). Since \( \succeq_{\mathcal{L}(M)} \) satisfies GSA with respect to \( \mathbb{R} \),

\[
\alpha F \circ m_p^{-1} \oplus (1 - \alpha) H \circ m_p^{-1} \succeq_{\mathcal{L}(M)} \alpha G \circ m_p^{-1} \oplus (1 - \alpha) H \circ m_p^{-1}
\]

\[
\therefore [\alpha F \oplus (1 - \alpha) H] \circ m_p^{-1} \succeq_{\mathcal{L}(M)} [\alpha G \oplus (1 - \alpha) H] \circ m_p^{-1}.
\]

Now by the construction of \( \succeq_{\mathcal{L}(M)} \),

\[
\alpha F \oplus (1 - \alpha) H \succeq \alpha G \oplus (1 - \alpha) H.
\]
This appendix provides an example of a preference relation for lotteries over a multi-commodity space that satisfies MGM but for which the money lottery preferences, induced directly from the preferences for lotteries over an income expansion path defined for some price system, do not satisfy Condition M.

Let \( V(F) = 2\left(\int x_{1}^{1/2} dF(x)\right) + \frac{2}{3}\left(\int x_{2}^{-3/2} dF(x)\right)^{-1} \) for \( F \in \mathcal{L}(X) \), where \( X = (0, 1] \times [1, \infty) \).

Hence \( V(\delta_{(x_{1}, x_{2})}) = 2 \cdot x_{1}^{1/2} + \frac{2}{3} \cdot x_{2}^{3/2} \) and so the income expansion path for prices \((1, 1)\) and incomes equal to or greater than 2 is the rectangular hyperbole \( x_{2} = x_{1}^{-1} \) (i.e., good 1 is inferior).

**B.1.** \( V(\cdot) \) satisfies MGM.

*Proof.* \( \forall F \in \mathcal{L}(X), \forall (y_{1}, y_{2}) \in X, \forall \alpha \in (0, 1] \)

(a) \( \frac{\partial}{\partial y_{1}} \left[ V([1 - \alpha]F \oplus \alpha \delta_{(y_{1}, y_{2})}) \right] = \alpha y_{1}^{-1/2} > 0 \)

(b) \( \frac{\partial}{\partial y_{2}} \left[ V([1 - \alpha]F \oplus \alpha \delta_{(y_{1}, y_{2})}) \right] = \alpha \cdot y_{2}^{-5/2} \left[ (1 - \alpha) \int x_{2}^{-3/2} dF(x) + \alpha y_{2}^{-3/2} \right]^{-2} > 0. \)

**B.2.** \( \succeq_{\mathcal{L}(M)} \) induced directly from \( V(\cdot) \) restricted to distributions whose support lies within \( Y(p) \) for \( p = (1, 1) \), does not satisfy Condition M.

*Proof.* \((1, 1)\) and \((\frac{1}{2}, 2)\) both lie on \( Y([1, 1]) \).

\[ V(\delta_{(1, 1)}) = \frac{5}{3} < \frac{2}{3} \cdot 2^{3/2} = V(\delta_{(0.5, 2)}) \]

\[ V([1 - \alpha] \delta_{(1, 1)} + \alpha \delta_{(0.5, 2)}) = 2[(1 - \alpha) + \alpha(0.5)^{0.5}] + \frac{2}{3}[(1 - \alpha) + \alpha 2^{-3/2}]^{-1} \]

but

\[ \frac{\partial V([1 - \alpha] \delta_{(1, 1)} + \alpha \delta_{(0.5, 2)})}{\partial \alpha} \bigg|_{\alpha = 0} = 2[(0.5)^{0.5} - 1] + \frac{2}{3} [1 - 2^{-3/2}] \]

\[ \simeq -0.155 < 0. \]

That is, starting with the degenerate lottery \( \delta_{(1, 1)} \), shifting a little probability weight onto the better bundle \((\frac{1}{2}, 2)\) decreases the utility of the lottery. This is a violation of AD1 and consequently a violation of Condition M for the induced money lottery preferences.
REFERENCES